High Explosive Detonation Propagation In Slab and Rate-Stick Geometries Near The Chapman-Jouguet Velocity

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1 Introduction

There has been significant recent work on understanding the variation of high explosive detonation phase velocity \(D_0\) in a two-dimensional slab geometric relative to that in an axisymmetric cylindrical (rate-stick) geometry having the same confinement as the slab. The ratio \(R(D_0)/T(D_0)\) has been termed the steady propagation scale factor by Jackson and Short [8], where \(R\) is the radius of a rate-stick that results in a given detonation phase velocity \(D_0\), while \(T\) is the corresponding thickness of a slab that result in the same detonation phase velocity. The ratio \(R(D_0)/T(D_0)\) varies as a function of \(D_0\). In the cylindrical rate-stick geometry, the detonation shock has two curvature components; the slab component which is the two-dimensional curvature along a diameter of the rate-stick, and the corresponding axisymmetric component. Petel et. al [9], Silvestrov et al. [10] and Higgins [6] have found a propagation scale factor \(R(D_0)/T(D_0) > 1\) for the explosives studied. In contrast, Jackson and Short [7, 8] found \(R(D_0)/T(D_0) < 1\) for three explosives nominally characteristic of ideal (PBX 9501), insensitive (PBX 9502) and non-ideal (ANFO) explosives. The purpose of the current work is to use a Detonation Shock Dynamics (DSD) model to give detailed insight into the dynamics behind the variation in the propagation scale factor \(R/T\) when the detonation phase velocity \(D_0\) approaches the Chapman-Jouguet velocity \(D_{CJ}\) for different degrees of confinement. In particular, we will extend the analysis in Jackson and Short [8] for larger variations in the difference between \(D_0\) and \(D_{CJ}\).

Detonation Shock Dynamics is an intrinsic surface propagation concept that replaces the detonation shock and reaction zone with a surface that evolves according to a prescribed intrinsic surface evolution law. Developed by Bdzil and Stewart [2–4, 11], it provides an advanced capability to describe detonation wave sweep through an arbitrarily complex geometry. At leading-order, the motion of the DSD surface relates the normal velocity of the surface \((D_n)\) to the local surface curvature \((\kappa)\), or

\[
D_n = f(\kappa).
\] (1)

The curvature \(\kappa\) represents the sum of the principal curvatures for any three-dimensional surface. For a given DSD form, determination of the detonation phase velocities in the slab and rate-stick geometries also requires information on how the HE is confined. This is done at the HE/material interface through specification of the “edge” angle, which we define here as the angle between the shock normal direction and the tangent to the material interface [5]. In Jackson and Short [8], it was shown that any detonation

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whose propagation can be adequately represented by \( D_0 \rightarrow D_{CJ} \) must necessarily have a scale factor \( R/T < 1 \), provided that, in the rate-stick geometry, the magnitude of the slab component of curvature increases monotonically with radius.

2 Formulation of Steady Detonation Propagation in Rate-Stick and Slab Geometries

Consider the steady propagation of an axisymmetric detonation in the positive axial \( z \) direction of a cylindrical explosive (rate-stick), where the DSD surface is given as a function of radial coordinate \( r \) by \( z = z_s(r) \), with a surface normal orientated in the direction of fresh reactants. Defining a level set function \( S = z - z_s(r) \), the normal to the surface is

\[
\mathbf{n} = \frac{\nabla S}{|\nabla S|} = \frac{1}{\sqrt{1 + [d(z_s(r))/dr]^2}} \left( -\frac{d}{dr} z_s(r) \mathbf{e_r} + \mathbf{e_z} \right),
\]

with a total curvature given by the sum of the slab (\( \kappa_s \)) and axisymmetric (\( \kappa_a \)) components, where

\[
\kappa = \nabla \cdot \mathbf{n} = \kappa_s + \kappa_a, \quad \kappa_s = -\frac{z_s''(r)}{(1 + [z_s'(r)]^2)^{3/2}}, \quad \kappa_a = -\frac{z_s'(r)}{r \left( 1 + [z_s'(r)]^2 \right)^{1/2}}.
\]

With \( D_0 \) as the steady axial detonation phase speed, the shock angle \( \phi \) between the axial direction and the surface normal \( \mathbf{n} \) at any point on the surface is determined by

\[
\cos \phi = \frac{D_n}{D_0} = \frac{1}{|\nabla S|} = \frac{1}{(1 + [z_s'(r)]^2)^{1/2}}, \quad \frac{dz_s}{dr} = -\tan \phi,
\]

so that \( \kappa \) can be written as

\[
\kappa = \kappa_s + \kappa_a, \quad \kappa_s = \frac{d\phi}{d\xi}, \quad \kappa_a = \frac{\sin \phi}{r},
\]

where \( \xi \) is surface arc length. Switching to \( \phi \) as the independent variable, the \((r, z)\) components of the surface shape can then be calculated by integration of

\[
\frac{dr}{d\phi} = \frac{\cos \phi}{\kappa_s}, \quad \frac{dz}{d\phi} = -\frac{\sin \phi}{\kappa_s},
\]

subject to

\[
z(\phi = 0) = 0, \quad r(\phi = 0) = 0, \quad \text{and } r(\phi = \phi_{edge}) = R,
\]

where \( \phi_{edge} \) is the shock angle at the edge of the explosive \((r = R)\). Due the \(1/r\) term in the axisymmetric curvature component in (5), the integration of (6) is started at a finite small value of \( \phi \), where

\[
r \sim \frac{\phi}{\kappa_s(\phi = 0)}, \quad z \sim -\frac{\phi^2}{2\kappa_s(\phi = 0)}, \quad \phi \ll 1,
\]

and on \( \phi = 0 \),

\[
D_n = D_0, \quad \kappa_s(\phi = 0) = \kappa_a(\phi = 0), \quad \kappa = 2\kappa_s(\phi = 0), \quad \text{where } \kappa_s(\phi = 0) = f(D_0)/2.
\]

For the 2D slab geometry, the above analysis is repeated, except that \( \kappa_s = 0 \), while \( r \) now refers to the distance from the slab center to the slab edge. Boundary conditions (7) with \( r(\phi = \phi_{edge}) = T/2 \), where \( T \) is the slab thickness, can be applied directly to the integration of (6).
3 Scaling Behaviour for $D_0 \to D_{CJ}$

We assume a $D_n - \kappa$ law of the linear form

$$\frac{D_n}{D_{CJ}} = 1 - B\kappa,$$

which allows the solutions in individual layers to be derived analytically. Note that the parameter $B$ represents a length scale that is characteristic of the reaction zone thickness. We are interested in the limit $D_0 \to D_{CJ}$, and define a small parameter $\epsilon$ such that

$$\epsilon = 1 - \frac{D_0}{D_{CJ}}, \quad \epsilon \ll 1.$$  

Slab Geometry: For the slab geometry, the differential equation (6) for $r(\phi)$ becomes

$$\left[1 - (1 - \epsilon) \cos \phi \right] \frac{d(r/B)}{d\phi} = \cos \phi,$$

subject to boundary conditions (7). For $\epsilon \ll 1$, we find an inner layer in the central part of the charge described by the scalings $\phi = \mathcal{O}(\sqrt{\epsilon})$ and $r/B = \mathcal{O}(1/\sqrt{\epsilon})$, where

$$\frac{r}{B} = \sqrt{2 \epsilon} \tan^{-1} \left( \frac{\phi}{\sqrt{2 \epsilon}} \right) + \mathcal{O}(\sqrt{\epsilon}).$$

If the degree of confinement is such that $\phi_{edge} = \mathcal{O}(\sqrt{\epsilon})$, (13) describes the solution from the charge center to the charge edge. Note that contained within the inner layer is a region of size $\phi = \mathcal{O}(\epsilon)$ around $r = 0$ where $r/B = \mathcal{O}(1)$, in which

$$\frac{r}{B} \sim \frac{\phi}{\epsilon}.$$  

For $\phi_{edge} = \mathcal{O}(1)$, an outer layer must be appended to the inner layer which extends to the edge of the charge. In this layer, $\phi = \mathcal{O}(1)$, $r/B = \mathcal{O}(1/\sqrt{\epsilon})$, where

$$\frac{r}{B} = \frac{\pi}{\sqrt{2 \epsilon}} - \phi - \frac{1}{\tan(\phi/2)} + \mathcal{O}(\sqrt{\epsilon}),$$

after matching with (13).

Rate-stick Geometry: For the rate-stick geometry, the differential equation (6) for $r(\phi)/B$ becomes

$$\left[1 - (1 - \epsilon) \cos \phi - \frac{\sin \phi}{(r/B)} \right] \frac{d(r/B)}{d\phi} = \cos \phi.$$  

We again find an inner layer in the central part of the charge where $\phi = \mathcal{O}(\sqrt{\epsilon})$ and $r/B = \mathcal{O}(1/\sqrt{\epsilon})$. The solution in this layer is

$$\frac{\phi}{\sqrt{\epsilon}} = \frac{\sqrt{2} J_1(\sqrt{\epsilon}r/\sqrt{2B})}{J_0(\sqrt{\epsilon}r/\sqrt{2B})},$$

where $J_0$ and $J_1$ are the order 0 and order 1 Bessel functions of the first kind. As for the slab geometry, contained within the inner layer is a region near $r = 0$ where $\phi = \mathcal{O}(\epsilon)$, in which

$$\frac{r}{B} \sim \frac{2\phi}{\epsilon}.$$
Case 2: Moderately strong confinement defined by detonation shock is small, which is consistent with this analysis. 

\[ \frac{r}{B} = \frac{\sqrt{2\beta}}{\sqrt{\epsilon}} - \phi - \frac{1}{\tan(\phi/2)} + O(\sqrt{\epsilon}), \]

where \( \beta \approx 2.40483 \) is the first positive zero of \( J_0(\beta) = 0 \).

**Scaling factor Implications.** The asymptotic analysis above reveals three cases of interest for the scaling factor ratio \( R/T \):

**Case 1:** Strong confinement defined by \( \phi_{\text{edge}} = O(1) \). In this case, there is a single layer describing the solution for \( 0 \leq \phi \leq \phi_{\text{edge}} \). In the rate-stick, the \( O(\epsilon) \) slab \( B\kappa_s \) and axisymmetric \( B\kappa_a \) components of curvature are equal across the charge. It then follows from (14) and (18) that

\[ 1 - \frac{R}{T} = O(\epsilon) > 0, \]

i.e. the scale factor is unity to leading-order for strong confinement defined by \( \phi_{\text{edge}} = O(1) \). Specifically, to \( O(\epsilon) \), it can be shown that \( R/T \sim 1 - \phi^2/12\epsilon \). Bdzil [1] has shown that the scale factor \( R(D_0)/T(D_0) = 1 \) can be approached in the limit where the streamline angle deflection behind the detonation shock is small, which is consistent with this analysis.

**Case 2:** Moderately strong confinement defined by \( \phi_{\text{edge}} = O(\sqrt{\epsilon}) \). In this case, there is again a single layer describing the solution for \( 0 \leq \phi \leq \phi_{\text{edge}} \). In the rate-stick, the scaled slab and axisymmetric curvature components are again of size \( O(\epsilon) \). However, in a region of this layer defined by \( O(\epsilon) < \phi \leq \phi_{\text{edge}} \), the two curvature components are no longer equal. This drives the scale factor below unity by \( O(1) \) amounts, i.e.

\[ 1 - \frac{R}{T} = O(1) > 0, \]

for moderately strong confinement defined by \( \phi_{\text{edge}} = O(\sqrt{\epsilon}) \). The actual value of the ratio \( R/T \) is determined through equations (13) and (17).

**Case 3:** Weak or no confinement defined by \( \phi_{\text{edge}} = O(1) \). In this case, the solution for \( 0 \leq \phi \leq \phi_{\text{edge}} \) is now described by two layers. The inner layer, represented by case 2 above, for \( \phi = O(\sqrt{\epsilon}) \), is joined to an outer layer where \( \phi = O(1) \). Significantly, in the outer layer, the curvature is dominated by the slab component where \( B\kappa_s = O(1) \), while \( B\kappa_a = O(\sqrt{\epsilon}) \). The outer layer solutions (15) and (19) show that in both cases the charge extent becomes independent of \( \phi \) to leading-order. Consequently, the scaling factor ratio \( R/T \) is constant to leading-order. Specifically, we find that

\[ \frac{R}{T} = \frac{\beta}{\pi} + O(\epsilon), \]

where \( \beta/\pi \approx 0.7655 \), for weak or no confinement defined by \( \phi_{\text{edge}} = O(1) \).

Figure [1] shows a comparison of the scale factor variation \( R/T \) with \( \phi \) derived from a composite of solutions (13) and (15) for the slab and (17) and (19) for the rate-stick (dashed line) and from numerical solutions of (6) (solid lines). A rapid decrease in the ratio of \( R/T \) is observed for small \( \phi \) before approaching close to the limit defined by (22). The agreement between the asymptotic and numerical solutions is excellent. Figure [2] shows a comparison of the scale factor \( R/T \) variation with \( D_0 \) derived from the composite asymptotic solutions and a numerical solution of (6) for an \( O(1) \) edge angle \( \phi_{\text{edge}} = 0.7033841 \). For small \( \epsilon \), the composite and numerical solutions are in good agreement. As \( \epsilon \) increases, the solutions diverge, but the asymptotic solutions still provide a reasonable approximation to the numerical solution even at values of \( D_0 \) significantly below \( D_{C,J} \) (at \( D_0 = 7 \text{ mm/\mu s} \), \( \epsilon = 0.0974 \)).
Figure 1: Comparison of the scale factor variation $R/T$ with $\phi$ derived from composite asymptotic solutions (dashed line) and from numerical solutions of (6) (solid lines). Here $B = 0.1$ cm and $D_{CJ} = 0.775525188$ cm/$\mu$s at a fixed phase velocity of $D_0 = 0.775$ cm/$\mu$s ($\epsilon = 6.772 \times 10^{-4}$). The composite and numerical solutions almost overlay in the plot.

Figure 2: Scale factor $R/T$ variation with changes in $D_0$ with $B = 0.1$ cm and $D_{CJ} = 0.775525188$ cm/$\mu$s for an edge angle $\phi_{edge} = 0.7033841$. A composite asymptotic solution (dashed line) and numerical solution of (6) (solid line) are shown.
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Asymptotics of DSD for $D_0 \rightarrow D_C$.

The agreement between the asymptotic and numerical solutions at larger $\varepsilon$ can be improved by extending the asymptotic analysis to an additional order. For instance, the inner slab solution for $\phi = \mathcal{O}(\sqrt{\varepsilon})$ is

$$
\frac{r}{B} \sim \sqrt{\frac{2}{\varepsilon}} \tan^{-1}\left(\frac{\phi}{\sqrt{2\varepsilon}}\right) + \sqrt{\varepsilon} \left(\frac{5\sqrt{2}}{4} \tan^{-1}\left(\frac{\phi}{\sqrt{2\varepsilon}}\right) - \frac{5\phi}{6\varepsilon} \left(\frac{\phi^2}{2} + 3\varepsilon\right)\right),
$$

(23)

while the outer slab solution for $\phi = \mathcal{O}(1)$ is

$$
\frac{r}{B} \sim \frac{\pi}{\sqrt{2\varepsilon}} - \phi - \frac{1}{\tan(\phi/2)} + \frac{5\sqrt{2}}{8} \sqrt{\varepsilon}.
$$

(24)

Similar extensions can be provided for rate-stick geometry, and the results used to generate a second-order accurate $R/T$ scaling factor variation with $D_0$.

References


