# Algebraic Methods for Robust Power Grid Analysis and Design

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# Overview

- What is a Power System?
- Classical Power System Problems
- Methods for Computing the Lyapunov Stability
- Positive Polynomials and Sum of Squares
- Methods for Computing the Region of Attraction
- Nonlinear System Decomposition

# Dynamic Systems Viewpoint

- A power system is a hybrid system characterized by:
  1. continuous and discrete states
  2. discrete events
  3. discrete dynamics
  4. mapping that define the evolution of discrete states.
- A power system is generically described by an indexed collections of DAEs:

$$
\begin{aligned}
\dot{x} &= f_u(x, y, \mu) \\
0 &= g_u(x, y, \mu)
\end{aligned}
$$

Reference: Ian Hiskens, *Power System Modeling for Inverse Problems*, TCS 51, 539-551, 2004.

# Dynamic Networks Viewpoint

A power system consists of generators and loads connected by transmission lines into a network structure.

- Traditional studies emphasize the modeling details of the node dynamics using simple network structures.
- More recent studies employ rather simple node dynamics and place more emphasis on network structure.

Reference: D. Hill and G. Chen, *Power Systems as Dynamic Networks*, ISCAS 2006.

# Control Systems Viewpoint

- The large scale system is represented as a collection of interconnected subsystems.
- Control problems are solved locally and then are combined with the interconnections to provide a global feedback law.
- It is difficult to identify the boundaries of the subsystems.
- Controlls are both local and remote.
- Time delays (usually random) are critical for the system's controllability.

Reference: A. I. Zecevic and D. D. Siljak, *Control of Complex Systems*, 2010.

# Classical Power System Problems

1. Using power flows to compute equilibria.
2. Applying static stability to check for voltage collapse phenomena (SNB).
3. Applying transient stability to check the stability of the operating point under external perturbations.
4. Studying (undamped) oscillations and instabilities (HB).

Note: These are various aspects of generic stability questions.

# Transient Stability Problem

- Assume an autonomous nonlinear system of the form

$$\dot{x} = f(x, \mu)\,, \tag{1}$$

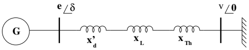
where $x \in \mathbb{R}^n$ and for which we assume $f(0, \mu) = 0$.

- We want to assess the stability of its equlibrium fixed point, $x_s = 0$, and to estimate its region of attraction:

$$A(0) = \{x \in \mathbb{R}^n : \lim_{t \to \infty} \Phi(x, t) = 0\} \tag{2}$$

# Example: One Machine Infinite Bus

- Consider this model:




$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 10\lambda - 20\sin(x_1) - x_2 \end{aligned}$$

- The equilibrium points can be found from the steady-state (power flow) equations:

$$\begin{aligned} 0 &= x_{20} \\ 0 &= 10\lambda - 20\sin(x_{10}) - x_{20} \end{aligned}$$

# Equilibria

- The solutions are:

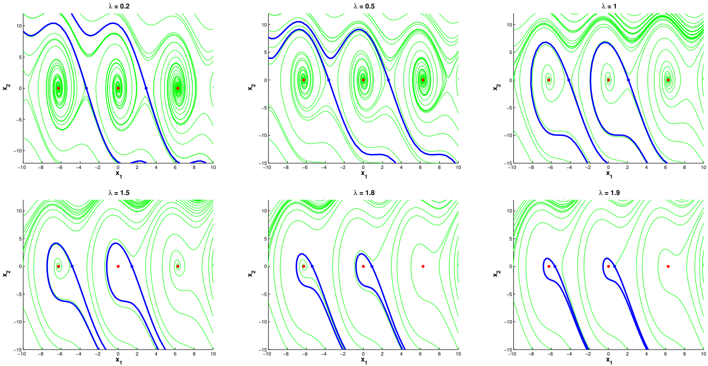
$$\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} \sin^{-1}(\lambda/2) \\ 0 \end{bmatrix} \tag{3}$$

- With two equilibrium points (and their periodic images):

$$x_{1s} = \sin^{-1}(\lambda/2)$$
$$x_{1u} = \pi - \sin^{-1}(\lambda/2)$$

Reference: Milano, F., *Power System Modelling and Scripting*, Springer, Heidelberg, in press.

# Stability and Region of Attraction



# Local Lyapunov Stability

**Theorem** For an open set $\mathcal{D} \subset \mathbb{R}^n$ with $0 \in \mathcal{D}$, suppose there exists a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ such that

$$\begin{aligned} V(0) &= 0\,, \\ V(x) &> 0 \qquad \forall z \in \mathcal{D}\,, \\ \frac{\partial V}{\partial z} f(z) &\leq 0 \qquad \forall z \in \mathcal{D}\,. \end{aligned}$$

Then $x = 0$ is a stable equilibrium point of (1). Any domain $\Omega_\beta := \{x \in \mathbb{R}^n \big| V(x) \leq \beta\}$ such that $\Omega_\beta \subseteq \mathcal{D}$ is a positively invariant region contained in the equilibrium point's ROA.

- Checking if $p \in \mathcal{R}_n$ is *positive semi-definite*, $p(x) \geq 0 \, \forall x$, is NP-hard when $\deg p \geq 4$.
- Replace this condition with a a polynomial-time *sufficient* condition for testing if $p$ is a sum of squares.
- $p$ is a *sum of squares* (SOS) if there exist polynomials $\{p_i\}_{i=1}^N$ such that $p = \sum_{i=1}^N p_i^2$.
- If $p$ is SOS then $p$ is PSD.

# Sums of Squares Polynomials

- **Theorem:** $p \in \mathrm{SOS}_{n,2d}$ iff there exists $Q \succeq 0$ and a vector of monomials $z_{n,d}$ such that $p = z_{n,d}^T Q z_{n,d}$, where

$$z_{n,d} := [1, x_1, x_2, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_n^2, \ldots, x_n^d]^T \qquad (4)$$

- All solutions to $p = z_{n,d}^T Q z_{n,d}$ can be expressed as $Q = Q_0 + \sum_{i=1}^h \lambda_i Q_i$ where $p = z_{n,d}^T Q_0 z_{n,d}$ and each $Q_i$ satisfies $z_{n,d}^T Q_i z_{n,d} = \theta$.

Reference: Parrilo, P., *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*, Caltech, 2000.

# Example

- The polynomial $p = 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4$ can be written as $p = z_{2,2}^T Q z_{2,2}$ where

$$z_{2,2} = \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}, Q_0 = \begin{bmatrix} 2 & 1 & -0.5 \\ 1 & 0 & 0 \\ -0.5 & 0 & 5 \end{bmatrix}, Q_1 = \begin{bmatrix} 0 & 0 & -0.5 \\ 0 & 1 & 0 \\ -0.5 & 0 & 0 \end{bmatrix}$$

- We can define an affine subspace of symmetric matrices related to $p$ as

$$\mathcal{S}_p = \{Q | z_{n,d}^T Q z_{n,d} = p(x)\} = \left\{ Q_0 + \sum_{i=1}^{h} \lambda_i Q_i \middle| \lambda_i \in \mathbb{R} \right\}$$

# SOS Example

- $p = 2x_1^4 + 2x_1^3 x_2 - x_1^2 x_2^2 + 5x_2^4$ is SOS since $Q_0 + \lambda_1 Q_1 \succeq 0$ for $\lambda_1 = 5$.
- An SOS decomposition can be constructed from a Cholesky factorization:

$$Q + \lambda_1 Q_1 = L^T L$$

  where:

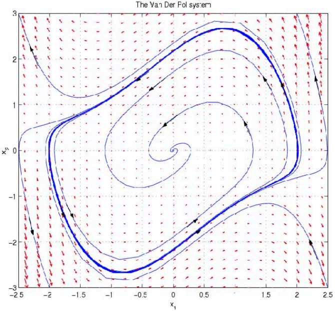
$$L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 1 & -3 \\ 0 & 3 & 1 \end{bmatrix}$$

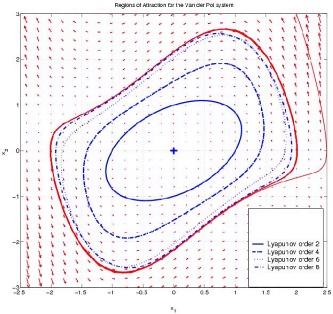
- Thus
  $p = (Lz)^T (Lz) == \frac{1}{2}(2x_1^2 - 3x_2^2 + x_1 x_2)^2 + \frac{1}{2}(x_3^2 + 3x_1 x_2)^2$

# Van der Pol Oscillator

$$\dot{x}_1 = -x_2$$

$$\dot{x}_2 = x1 - (1 - x_1^2)x_2$$



# Computational Implementation

- The SOS conditions are feasibility conditions for LMI.
- They can be converted into appropriate SDP conditions using SOSTOOLS.
- SOSTOOLS then calls a SDP solver (SeDuMi) and then converts the solution back to the original SOS program.

Reference: A. Papachristodoulou and S. Prajan, *a tutorial on sum of squares techniques for system analysis*, CDC 2005.

# Methodology for Non-polynomial vector fields

- Consider again the one-machine infinite-bus system:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= 10\lambda(1-\cos(x_1)) - 20\cos(x_{1s})\sin(x_1) - x_2\end{aligned}$$

- Define $x_3 = \sin(x_1)$ and $x_4 = 1 - \cos(x_1)$.

$$\dot{x}_1 = x_2 \tag{5}$$
$$\dot{x}_2 = 10\lambda x_4 - 20\cos(x_{1s})x_3 - x_2 \tag{6}$$
$$\dot{x}_3 = (1 - x_4)x_2 \tag{7}$$
$$\dot{x}_4 = x_3 x_2 \tag{8}$$

and introduce an equality constraint $x_3^2 + (1 - x_4)^2 = 1$ .

- Generally, for a non-polynomial system $\dot{x} = f(x, \mu)$ the recasted system is written as:

$$\dot{\tilde{x}}_1 = f_1(\tilde{x}_1, \tilde{x}_2)\,,$$
$$\dot{\tilde{x}}_2 = f_2(\tilde{x}_1, \tilde{x}_2)\,,$$

  where $\tilde{x}_1 = (x_1, \ldots, x_n) = z$ are the original state variables, $\tilde{x}_2 = (x_{n+1}, \ldots, x_{n+m}) = F(\tilde{x}_1)$ are the new variables.
- The recasting process introduces constraints:

$$G(\tilde{x}_1, \tilde{x}_2) = 0 \tag{9}$$

# Extension of Lyapunov Stability Theorem

- Let $\mathcal{D}_1 \subset \mathbb{R}^n$ and $\mathcal{D}_2 \subset \mathbb{R}^m$ be open sets such that $0 \in \mathcal{D}_1$ and $F(\mathcal{D}_1) \subseteq \mathcal{D}_2$.
- Assume that $\mathcal{D}_1 \times \mathcal{D}_2$ is a semialgebraic set defined by the following inequalities:

$$\mathcal{D}_1 \times \mathcal{D}_2 = \{(\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^n \times \mathbb{R}^m : G_{\mathcal{D}}(\tilde{x}_1, \tilde{x}_2) \geq 0\}\,.$$

Reference: Papachristodoulou, A. and Prajna, S., *Analysis of Non-polynomial systems Using the Sum of Squares Decomposition*, Positive Polynomials in Control, pp. 23-43, 2005.

# Proposition

Suppose that for the system (5) and the functions $F(\tilde{x}_1)$, $G_1(\tilde{x}_1, \tilde{x}_2)$, $G_2(\tilde{x}_1, \tilde{x}_2)$, and $G_{\mathcal{D}}(\tilde{x}_1, \tilde{x}_2)$ there exists polynomial functions $\lambda_{1,2}(\tilde{x}_1, \tilde{x}_2)$, and SOS polynomials $\sigma_{1,2}(\tilde{x}_1, \tilde{x}_2)$, such that

$$
\begin{aligned}
& V(0, \tilde{x}_{2,0}) = 0\,, \\
& V - \lambda_1^T G - \sigma_1^T G_{\mathcal{D}} - \phi \in \Sigma_n\,, \\
& -\left(\frac{\partial V}{\partial \tilde{x}_1} f_1 + \frac{\partial V}{\partial \tilde{x}_2} f_2\right) - \lambda_2^T G - \sigma_2^T G_{\mathcal{D}} \in \Sigma_n\,,
\end{aligned}
$$

where $\phi(\tilde{x}_1, F(\tilde{x}_2)) > 0$ for $\forall \tilde{x}_1 \in \mathcal{D}_1 \setminus 0$, then $z = 0$ is a stable equilibrium of (1).

- Define an equality constraint: $G := x_3^2 + x_4^2 - 2x_4$.
- Define $\mathcal{D}_1 \times \mathcal{D}_2$ as:

$$G_{\mathcal{D}}(1) = \beta^2 - (x_1^2 + x_2^2) \geq 0$$
$$G_{\mathcal{D}}(2) = (x_3 - \sin(\beta))(x_3 + \sin(\beta)) \geq 0$$

- Define $\phi(\tilde{x}_1, \tilde{x}_2) = \sum_{i=1}^{4} \epsilon_i x_i^2$ with $\epsilon_i \geq 0$.

# Example: One Machine Infinite Bus System

- Solve the following optimization problem:

$$\max_{\epsilon,\lambda\in\mathcal{R}_4,\sigma\in\Sigma_4} \beta$$

$$\begin{aligned}\text{subject to:} \qquad & V-\lambda_1 G-\sigma_1 G_{\mathcal{D}}(1)-\sigma_2 G_{\mathcal{D}}(2)-\phi\succeq 0\\ & -\frac{dV}{dt}-\lambda_2 G-\sigma_3 G_{\mathcal{D}}(1)-\sigma_4 G_{\mathcal{D}}(2)\succeq 0\end{aligned}$$

$$\begin{aligned}V = {} & 0.0020275x_1^2-0.0042255x_1\sin(x_1)-0.04157x_1(1-\cos(x_1))\\ & -0.0001238x_1+0.014573x_2^2+0.0029823x_2\sin(x_1)\\ & -0.00034485x_2(1-\cos(x_1))+0.20613\sin(x_1)^2\\ & +0.016014\sin(x_1)(1-\cos(x_1))+0.2033(1-\cos(x_1))^2\\ & +0.17784(1-\cos(x_1))\end{aligned}$$

# Energy Functions Approach

- Numerical integration methods derive the fault on trajectory.
- Time-domain methods for transient stability analysis calculate the post fault behavior via numerical integration.
- Direct methods determine, based on energy functions, whether the initial point of the postfault trrajectory lies inside the ROA of the stable equlibirum point.
- Direct methods can handle large power systems and include excitation controls.

Reference: H.D. Chiang, *Direct Methods for Stability Analysis of Electric Power Systems*, 2011.

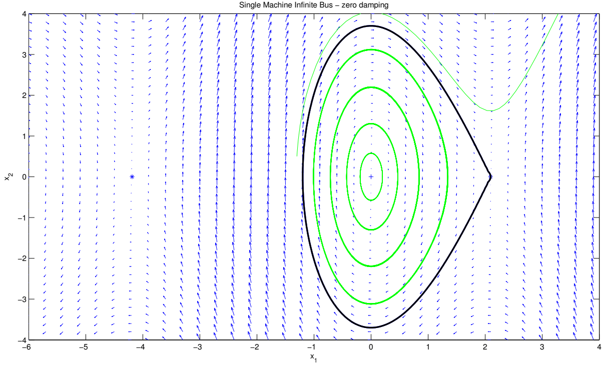
- There exists an energy function for the OMIB system:

$$W = 0.1x_2^2 - x_1 + \sin x_3 + \sqrt{3}x_4$$

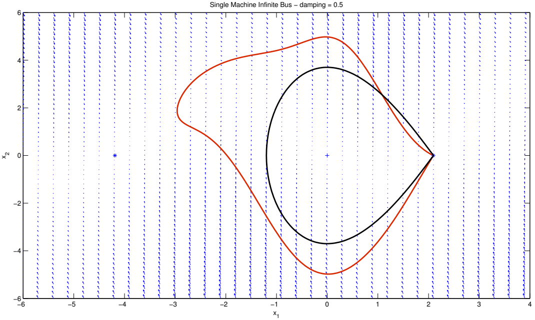
- This is an energy function for the undamped system!
- There are no energy functions for damped power systems.

# Zero Damping

Under simplifying assumptions we can construct the energy function from our Lyapunov function.



# Zero Damping Limit is Conservative




- With SOS methods the ROA can increase with damping.
- There is an energy function for the dissipative OMIB system.

# Remarks

SOS methods can be extended to analyze:

- Switched and hybrid systems.
- Systems with time delays.
- Systems with parametric uncertainties. We have constructed Lyapunov functions parameterized by unknown damping.
- Robust bifurcation analysis using semi-algebraic set descriptions.

- The size of the SDP depends on: 1) the dimension of the state space; 2) the order of the vector field and 3) the order of the Lyapunov functions.
- It is difficult to construct Lyapunov functions of systems with state dimension larger than 6, for cubic vector fields and quartic Lyapunov functions.
- We can make the computation of SOS problems more scalable by structuring the Lyapunov functions appropriately.
- The polynomial expression become sparse and sparsity algorithms can be used to find the SOS decomposition.

Reference: P. Parrilo, , in *Positive Polynomials in Control*, 2005.

# Nonlinear Composite Lyapunov functions

- Construct a weighted graph based on the energy flows between states of the original system.
- Patition the state space into subgraphs $x = (x_1, \dots, x_k)$ which minimize the energy flows between partitions:

$$\dot{x}_1 = f_1(x_1) + g_1(x_1, u_1), u_1 = x_2 \quad (10)$$

$$\dot{x}_2 = f_2(x_2) + g_2(x_2, u_2), u_2 = x_1 \quad (11)$$

- Use SOS methods to construct $V_i(x_i)$ such that:

$$V_1(x_1) > 0, -\frac{\partial V_1}{\partial x_1} f_1(x_1) > 0, V_2(x_2) > 0, -\frac{\partial V_2}{\partial x_2} f_2(x_2) > 0.$$

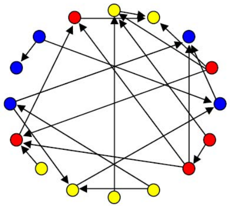
- Define a composite Lyapunov function $V_c(x) = \sum \alpha_i V_i(x_i)$ where $\alpha_i$ are found such that $-\frac{\partial V_c}{\partial x} f(x) > 0$.

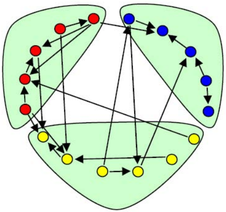
# Numerical Example

- 16 state non-linear system with second order dynamics:

$$\dot{x}_i = x_i(b_i - x_i - \sum_{j=1}^{n} A_{ij}x_j)$$

- Direct SOS analysis is not possible.





# Conclusions

- SOS methods can be used to generalize energy function methods for stability analysis.
- The technique can be combined with decomposition approaches for stability analysis of large scale systems.
- It can handle a large class of systems, especially with heterogeneous dynamics.