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Time-dependent isotropic turbulence*

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Abstract. Homogeneous isotropic turbulence subject to linearly increasing forcing is investigated as a unit problem for statistically unsteady turbulence. The transient spectral dynamics is analysed using a closure theory. A long time asymptotic state is found with $k^{-7/3}$ corrections to the energy spectrum as proposed by Yoshizawa. Although the cancellation of $\mathrm{O}(Re^{1/2})$ terms underlying the standard dissipation rate equation is confirmed in this asymptotic state, it is found that this equation cannot predict the transient dynamics accurately. The discrepancies are explained in terms of the basic mechanisms of vortex stretching and enstrophy destruction responsible for the evolution of the dissipation rate.

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We would like to dedicate this paper to Professor Akira Yoshizawa on the occasion of his retirement from the University of Tokyo: gotaikan omedetou gozaimasu.

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Contents

1. Introduction

Recent technological developments [1] have called attention to the importance of predicting statistically unsteady turbulent flows. Isotropic turbulence driven by unsteady forcing defines a class of problems suitable for testing ideas about modelling these flows. The absence of linear mechanisms is appropriate because many technologically important unsteady flows are subject to at most weak, unsteady shearing. These problems also provide good test cases for the ϵ equation because their dynamics is dominated entirely by non-linearity.

This paper investigates the special problem of homogeneous, isotropic turbulence, initially in a steady state, driven by forcing with a linearly growing amplitude. Although admittedly a model problem, it was chosen because the imposed non-stationarity is as simple as possible. Therefore, on the one hand, linearly increasing forcing provides a problem in which the two-equation model is most likely to be successful and, on the other hand, a good place to analyse deficiencies of the two-equation model if it fails. The spectral dynamics is studied using a recently developed spectral closure [2] that is intermediate in complexity between the eddy-damped quasi-normal Markovian (EDQNM) approximation and the Heisenberg model and closely related to the model of Canuto and Dubovikov [3]. The transient and asymptotic predictions of this closure are compared with the predictions of an appropriate two-equation model.

At long times, an asymptotic state develops in which the kinetic energy increases with time as $K \sim t^{2/3}$ and the production to dissipation ratio, $P/\epsilon \sim 1$. This state is characterized to leading order by a growing Kolmogorov spectrum, but with the $k^{-7/3}$ corrections proposed by Yoshizawa [4] for time-dependent turbulence. The cancellation of the $\mathrm{O}(Re^{1/2})$ vortex stretching and enstrophy destruction terms required [5] to formulate the standard ϵ equation is confirmed analytically; the $\mathrm{O}(Re^0)$ remainder is determined by Yoshizawa's $k^{-7/3}$ correction term.

It is shown that this asymptotic state can be described by a two-equation model with non-standard values of the constants $C_{\epsilon 1}$ and $C_{\epsilon 2}$. The conclusion that an asymptotic state can be described by a two-equation model agrees with previous analyses [6, 7] and underlies our choice of this problem. However, the transient evolution is not predicted accurately. The discrepancy is analysed using the closure theory, which provides access to the details of the dynamics of ϵ.

As was just noted, the dissipation rate transport equation requires a balance between two terms proportional to vortex stretching and enstrophy destruction [5, 8]. Two-equation modelling simply asserts that this balance can be modelled in terms of production, kinetic energy and the dissipation rate itself. There is no fundamental justification for this assertion: there is simply no alternative if a model is to be formulated using these basic quantities alone.

The choice of these dynamic descriptors makes two problems unavoidable: first, because they are linked to relatively small scales of motion, the onset of non-trivial dynamics of vortex stretching and enstrophy destruction will occur later than the imbalance of production and dissipation; and secondly, the establishment of a balance between vortex stretching and enstrophy destruction occurs on the rapid time scales related to these small scales of motion, not on the integral time scale of the two-equation model. The consequences are that a two-equation model will predict that ϵ begins to grow both sooner and more gradually than it should. Both effects are confirmed numerically by closure computations and direct numerical simulation (DNS).

The conclusion that rapid time scales can dominate the transient evolution of ϵ can be compared with the proposal of Speziale and Bernard [8] that the Kolmogorov time scale replaces the integral time scale in the ϵ equation, thereby introducing explicit Reynolds number dependence into this equation. This very unorthodox proposal was motivated by a dynamic picture of *unbalanced vortex stretching*, in which the $\mathrm{O}(Re^0)$ balance between vortex stretching and enstrophy destruction envisioned in the standard discussions of the ϵ equation does not occur.

Unlike Speziale and Bernard, we find that the possibility of unbalanced vortex stretching is limited to the transient evolution. Furthermore, it is far from certain that it is the Kolmogorov time scale that is relevant: we find far weaker Reynolds number dependence than dependence on this time scale would produce. Finally, at long times, the $\mathrm{O}(Re^0)$ balance is established, leading to an asymptotic state that can be correctly described by a two-equation model of the standard form.

Nevertheless, the essential point made by Speziale and Bernard that the dynamics of the dissipation rate is not necessarily governed by the integral time scale, is confirmed in this paper. On intuitive grounds, this conclusion is not surprising: what is surprising is that the dynamics of the dissipation rate ever actually is governed by the integral time scale: this possibility is linked to the idea that in 'equilibrium' turbulence, the dissipation rate is scale-independent and, hence, a property of both large and small scales.

That the long time asymptotic dynamics is governed by the integral scale is to some extent geometric or kinematic: in the asymptotic state, the two-equation model and governing equations share certain common symmetry or invariance properties which greatly restrict the number of possible length and time scales [6, 9]. The role of Yoshizawa's correction term gives these invariance properties an additional dynamic significance since it proves to be most significant at the large scales of motion. In summary, this paper finds conditions under which the ϵ equation can be valid and delineates its theoretical limitations in a special problem of time-dependent turbulence.

2. Statement of the problem

Consider homogeneous isotropic turbulence in a Kolmogorov steady state that has developed under steady random forcing. The steady-state spectral evolution equation is [10]

$$0 = \Pi(k/k_0) - S(k) - 2\nu k^2 E(k), \tag{1}$$

where $E(k)$ is the energy spectrum, $S(k)$ the transfer spectrum and the production spectrum $\Pi(\kappa)$ peaks at $\kappa = 1$.

Starting from the initial condition defined by equation (1), increase the force amplitude linearly, so that the spectral evolution equation becomes

$$\dot{E}(k,t) = (1 + t/\tau)\Pi(k/k_0) - S(k,t) - 2\nu k^2 E(k,t). \tag{2}$$

It will be convenient to call this turbulent flow *ramp flow*, and the constant τ^{-1} the *ramp rate*. In this problem, the forcing scale k_0 is fixed; only the force amplitude changes with time. Define total production $P(t)$ by

$$P(t) = (1 + t/\tau)\int_0^\infty \Pi(k/k_0)\,\mathrm{d}k. \tag{3}$$

The energy equation, obtained by integrating equation (2) over all wavenumbers k, is

$$\dot{K} = P - \epsilon, \tag{4}$$

where, as usual,

$$K = \int_0^\infty E(k,t)\,\mathrm{d}k, \qquad \epsilon = 2\nu\int_0^\infty k^2 E(k,t)\,\mathrm{d}k \tag{5}$$

and conservation of energy by non-linear interaction implies

$$\int_0^\infty S(k,t)\,\mathrm{d}k = 0. \tag{6}$$

3. Elementary scaling analysis

We first consider a possible asymptotic state for this problem. Although it is probable that P will always exceed ϵ, the ratio P/ϵ might approach a constant. Assume then that $\epsilon \sim t$. Since by hypothesis the integral scale does not change, $K^{3/2}/\epsilon$ should be constant. Since $\epsilon \sim t, K \sim t^{2/3}$. However, then the kinetic energy equation (4) requires $P - \epsilon \sim t^{-1/3}$. In particular, $P/\epsilon \to 1$. To summarize, in this asymptotic state, the total production, dissipation and kinetic energy scale with time as

$$P \sim t, \qquad \epsilon \sim t - at^{-1/3}, \qquad K \sim t^{2/3} \tag{7}$$

for some constant a. Thus, ramp flow transitions from a simple Kolmogorov steady state to another simple but time-dependent asymptotic state. The transient dynamics through which turbulence reorganizes itself between these states proves to be non-trivial.

Since it is characterized by certain constant limiting ratios, in this case of $K^{3/2}/\epsilon$ and P/ϵ, such an asymptotic state is sometimes called a *fixed point*: its existence reflects certain scaling invariance properties and consequent similarity solutions of the Navier–Stokes equations. This viewpoint can be developed in much greater depth [9].

4. Closure analysis of the asymptotic state

Ramp flow will be analysed using the recently developed Cartoon model of spectral behaviour (CMSB) closure [2]. This model closes the energy transfer term in equation (2) by

$$S(k) = \frac{\partial \mathcal{F}}{\partial k}, \tag{8}$$

where the energy flux $\mathcal{F}$ is

$$\mathcal{F}(k) = c\left\{\int_0^k \mathrm{d}\kappa\,\kappa^2 E(\kappa)\int_k^\infty \mathrm{d}p\,\theta(p)E(p) - \int_0^k \mathrm{d}\kappa\,\kappa^4\int_k^\infty \mathrm{d}p\,\theta(p)\frac{E(p)^2}{p^2}\right.$$
$$\left. +\beta\left[\int_0^k \mathrm{d}\kappa\,\kappa^2 E(\kappa)\int_k^\infty \mathrm{d}p\,\theta(p)p\frac{\mathrm{d}E}{\mathrm{d}p} - \int_0^k \mathrm{d}\kappa\,\kappa^4\int_k^\infty \mathrm{d}p\,\theta(p)p\frac{\mathrm{d}E}{\mathrm{d}p}\frac{E(p)}{p^2}\right]\right\}. \qquad (9)$$

The time-scale θ satisfies the evolution equation

$$\dot{\theta}(k) = 1 - \eta(k)\theta(k) - \nu k^2\theta(k), \qquad (10)$$

where the damping factor η is

$$\eta(k) = c_\eta\theta(k)\int_0^k \mathrm{d}p\,p^2E(p). \qquad (11)$$

Time arguments are assumed without being explicitly written in equations (9)–(11). Suggested values for the constants are discussed in [2].

This closure results from a drastic simplification of the non-linear interactions in closures based on the direct interaction approximation [11] which nevertheless retains some key properties of such closures, including the importance of distant interactions and energy backscatter. The advantage of this kind of closure is its computational simplicity (cf also the local wave-number (LWN) closure [7, 6]). The first term on the right-hand side of equation (9) corresponds to the classical Heisenberg model [10]. It provides eddy damping of the excitation at mode k. The second term provides energy input to mode k and may be considered to represent energy 'backscatter'. This combination of terms was proposed by Canuto and Dubovikov [3]; the present model only differs in the third and fourth terms on the right-hand side of equation (9) and the use of time-scale evolution equations (10) and (11).

Next, we refine the elementary scaling arguments of section 3 by extending them to the closure equations. If the solution exhibits a range of scales satisfying Kolmogorov relations

$$E(k) = C_K\epsilon^{2/3}k^{-5/3}, \qquad \eta(k) = C_D\epsilon^{1/3}k^{2/3}, \qquad (12)$$

then, since k_0 is fixed, we can anticipate the asymptotic time dependence

$$E \sim t^{2/3}, \qquad \eta \sim t^{1/3} \qquad (13)$$

and equations (11) and (9) imply

$$\theta \sim t^{-1/3}, \qquad \mathcal{F} \sim t^1. \qquad (14)$$

It is clear, however, that equation (2) with the closure relations (8) and (9) will not be satisfied by spectral quantities with these exact scalings. Instead, the correction to the scaling of ϵ in equation (7) suggests expansions

$$E(k,t) = (t/\tau)^{2/3}E_0(k) + (t/\tau)^{-2/3}E_1(k) + \cdots,$$
$$\theta(k,t) = (t/\tau)^{-1/3}\theta_0(k) + (t/\tau)^{-5/3}\theta_1(k) + \cdots \qquad (15)$$

of the spectral quantities in powers of $(t/\tau)^{-4/3}$. Equating coefficients of like powers of t/τ gives

$$\frac{\partial\mathcal{F}_0}{\partial k} = 0, \qquad 1 - \theta_0^2\int_0^k \mathrm{d}\kappa\,\kappa^2E_0(\kappa) = 0, \qquad (16)$$

where

$$\mathcal{F}_0(k) = c\left\{\int_0^k \mathrm{d}\kappa\,\kappa^2 E_0(\kappa)\int_k^\infty \mathrm{d}p\,\theta_0(p)E_0(p) - \int_0^k \mathrm{d}\kappa\,\kappa^4\int_k^\infty \mathrm{d}p\,\theta_0(p)\frac{E_0(p)^2}{p^2}\right.$$
$$\left.+\beta\left[\int_0^k \mathrm{d}\kappa\,\kappa^2 E_0(\kappa)\int_k^\infty \mathrm{d}p\,\theta_0(p)p\frac{\mathrm{d}E_0}{\mathrm{d}p} - \int_0^k \mathrm{d}\kappa\,\kappa^4\int_k^\infty \mathrm{d}p\,\theta_0(p)p\frac{\mathrm{d}E_0}{\mathrm{d}p}\frac{E_0(p)}{p^2}\right]\right\} \tag{17}$$

to leading order. At the next order,

$$\frac{2}{3}E_0 = \frac{\partial\mathcal{F}_1}{\partial k}, \qquad \frac{1}{3} = \theta_0\int_0^k \mathrm{d}\kappa\,\kappa^2 E_1(\kappa) + 2\theta_1\int_0^k \mathrm{d}\kappa\,\kappa^2 E_0(\kappa), \tag{18}$$

where

$$\mathcal{F}_1(k) = c\left\{\int_0^k \mathrm{d}\kappa\,\kappa^2 E_1(\kappa)\int_k^\infty \mathrm{d}p\,\theta_0(p)E_0(p) + \int_0^k \mathrm{d}\kappa\,\kappa^2 E_0(\kappa)\int_k^\infty \mathrm{d}p\,\theta_1(p)E_0(p)\right.$$
$$+\int_0^k \mathrm{d}\kappa\,\kappa^2 E_0(\kappa)\int_k^\infty \mathrm{d}p\,\theta_0(p)E_1(p) - \int_0^k \mathrm{d}\kappa\,\kappa^4\int_k^\infty \mathrm{d}p\,\theta_1(p)\frac{E_0(p)^2}{p^2}$$
$$-\int_0^k \mathrm{d}\kappa\,2\kappa^4\int_k^\infty \mathrm{d}p\,\theta_0(p)\frac{E_0(p)E_1(p)}{p^2} + \beta\left[\int_0^k \mathrm{d}\kappa\,\kappa^2 E_1(\kappa)\int_k^\infty \mathrm{d}p\,\theta_0(p)p\frac{\mathrm{d}E_0}{\mathrm{d}p}\right.$$
$$+\int_0^k \mathrm{d}\kappa\,\kappa^2 E_0(\kappa)\int_k^\infty \mathrm{d}p\,\theta_1(p)p\frac{\mathrm{d}E_0}{\mathrm{d}p} + \int_0^k \mathrm{d}\kappa\,\kappa^2 E_0(\kappa)\int_k^\infty \mathrm{d}p\,\theta_0(p)p\frac{\mathrm{d}E_1}{\mathrm{d}p}$$
$$-\int_0^k \mathrm{d}\kappa\,\kappa^4\int_k^\infty \mathrm{d}p\,\theta_1(p)p\frac{\mathrm{d}E_0}{\mathrm{d}p}\frac{E_0(p)}{p^2} - \int_0^k \mathrm{d}\kappa\,\kappa^4\int_k^\infty \mathrm{d}p\,\theta_0(p)p\frac{\mathrm{d}E_1}{\mathrm{d}p}\frac{E_0(p)}{p^2}$$
$$\left.\left.-\int_0^k \mathrm{d}\kappa\,\kappa^4\int_k^\infty \mathrm{d}p\,\theta_0(p)p\frac{\mathrm{d}E_0}{\mathrm{d}p}\frac{E_1(p)}{p^2}\right]\right\}. \tag{19}$$

Equations (16) and (17) coincide with the balance equations for a Kolmogorov steady state. Therefore, to leading order, the flux is independent of k, even in this time-dependent case and E_0 and θ_0 must be consistent with a Kolmogorov steady state; of course, equation (15) states that E and θ are indeed time-dependent, and that the spectral flux increases with time.

Equations (18) represent a linear system of integral equations for E_1 and θ_1. Rather than writing the lengthy system of equations explicitly, we simply note that it is consistent with correction terms

$$E_1 \sim k^{-7/3}, \qquad \theta_1 \sim k^{-4/3}, \tag{20}$$

which evidently formally balance all terms in equation (18). The $k^{-7/3}$ scaling of the energy spectrum correction is consistent with the analysis of non-equilibrium turbulence of Yoshizawa [4]. In the context of Yoshizawa's analysis, it should be noted that the non-equilibrium time scales $P/\dot{P} \sim \epsilon/\dot{\epsilon} \sim t$ are very large; consequently, we are in the regime of weak time dependence in which perturbation about Kolmogorov scaling is appropriate.

5. Two-equation modelling

The exact ϵ equation in spectral form is

$$\dot{\epsilon} = P_2 + S - G, \tag{21}$$

where

$$P_2 = 2\nu \int_0^\infty \mathrm{d}k\, k^2 P(k/k_0), \qquad S = -2\nu \int_0^\infty \mathrm{d}k\, k^2 \frac{\partial \mathcal{F}}{\mathrm{d}k}, \qquad G = -4\nu^2 \int_0^\infty \mathrm{d}k\, k^4 E(k). \tag{22}$$

S is ν times the *vortex stretching* term in the entrophy balance equation, and G is ν times the *enstrophy destruction*. Since the production spectrum peaks at large scales, P_2 is typically small, and in a steady state, $S = G$, which is equivalent to Batchelor's [10] relation for the skewness in a Kolmogorov steady state in which $k_0 \to 0$.

Assuming a steady-state Kolmogorov spectrum with high-wavenumber cutoff at $k_d \sim (\epsilon/\nu^3)^{1/4}$ so that $\nu \sim k_d^{-4/3}$,

$$G \sim \nu^2 k_d^{10/3} = k_d^{2/3} \sim Re^{1/2}. \tag{23}$$

Since

$$S = 4\nu \int_0^\infty \mathrm{d}k\, k\mathcal{F}(k) \tag{24}$$

and in the inertial range, $\mathcal{F}(k)$ is constant,

$$S \sim \nu k_d^2 \sim k_d^{2/3} \sim Re^{1/2}. \tag{25}$$

Thus, in a Kolmogorov steady state, with $P_2 \approx 0$, both terms on the right-hand side of equation (21) are of order $\mathrm{O}(Re^{1/2})$, but in the asymptotic limit of $k_0 \to 0$, their sum is zero.

The formulation of a dissipation rate transport equation poses a fundamental question about turbulence dynamics: the classic statement of this question by Tennekes and Lumley [5] will be reviewed briefly. A basic premise of turbulence theory is that the dynamics of low-order moments in high Reynolds number turbulence is in fact independent of Reynolds number. If valid, this premise requires that the right-hand side of equation (21) be independent of Reynolds number. Therefore, although equations (23) and (25) suggest that both S and G in equation (21) will be of order $\mathrm{O}(Re^{1/2})$ even in a time-dependent flow, the difference $S - G$ must nevertheless be of the order $\mathrm{O}(Re^0)$.

It has been difficult to give a completely convincing theoretical argument to justify this hypothesis. Indeed, the absence of such arguments led Speziale and Bernard [8] to propose that this cancellation of divergences does not occur, leading to a model in which $\dot{\epsilon}$ is proportional to the Kolmogorov frequency scale $\sqrt{\epsilon/\nu}$. The consequent $\nu^{-1/2} \sim \mathrm{O}(Re^{1/2})$ imbalance on the right-hand side of equation (21) was applied by Speziale and Bernard to decaying turbulence [8] and to homogeneous shear flow [12] leading to novel predictions about both flows.

In the present problem of ramp flow, expand S and G in series

$$S = S_0 + S_1 + \cdots, \qquad G = G_0 + G_1 + \cdots, \tag{26}$$

where

$$S_0 = -2\nu \int_0^\infty \mathrm{d}k\, k^2 \frac{\partial \mathcal{F}_0}{\mathrm{d}k}, \qquad S_1 = -2\nu \int_0^\infty \mathrm{d}k\, k^2 \frac{\partial \mathcal{F}_1}{\mathrm{d}k} \tag{27}$$

and

$$G_0 = -4\nu^2 \int_0^\infty \mathrm{d}k\, k^4 E_0(k), \qquad G_1 = -4\nu^2 \int_0^\infty \mathrm{d}k\, k^4 E_1(k). \tag{28}$$

In equations (27), $\mathcal{F}_0$ and $\mathcal{F}_1$ are defined by equations (17) and (19). The analysis in section 4 shows that despite the time dependence, the leading-order terms E_0 and θ_0 in equation (15) are simply Kolmogorov. This implies that $S_0 = G_0$ even in this time-dependent problem and the divergences of order $\mathrm{O}(Re^{1/2})$ do exactly cancel.

The remainder is evaluated using the spectral corrections found in equation (20). First,

$$G_1 \sim \nu^2 \int_0^\infty \mathrm{d}k\, k^4 E_1(k) \sim \nu^2 k_d^{8/3} \sim k_d^0 \tag{29}$$

is finite at high wavenumbers. Similarly, considering only a typical term in $\mathcal{F}_1$,

$$\nu \int_0^\infty \mathrm{d}k\, k \int_0^k \mathrm{d}\kappa\, \kappa^2 E_1(\kappa) \int_k^\infty \mathrm{d}p\, \theta_0(p) E_0(p) \sim \nu k_d^{4/3} \sim k_d^0. \tag{30}$$

The same conclusion obtains for the other contributions to S_1. It follows that

$$S - G = S_1 - G_1 = \mathrm{O}(Re^0) \tag{31}$$

in the asymptotic regime.

Although equation (31) confirms the formulation of Tennekes and Lumley [5], it is not a useful guide to formulating the ϵ equation: it rules out the explicit appearance of viscosity without suggesting any particular analytical form. Two-equation modelling simply proposes to close the difference $S - G$ in terms of the basic single-point moments P, K, and ϵ themselves. We repeat that it has never been claimed that this proposal has any fundamental justification; there is simply no alternative if the model is to be closed in terms of these quantities alone.

In the case of ramp flow, adding to the usual dimensional arguments the requirement that the two-equation model be consistent with a Kolmogorov steady state, the initial conditions for ramp flow, the ϵ equation must have the form

$$\dot{\epsilon} = C \frac{\epsilon}{K}(P - \epsilon) \tag{32}$$

so that, in the standard notation for the constants in two-equation models, $C_{\epsilon 1} = C_{\epsilon 2}$. Note that Yoshizawa [13] found this (supposedly incorrect) result by a general statistical argument, and that it is also cited by Dejoan and Schiestel [14] as appropriate for models of small-scale dynamics in multiple-scale modelling.

In the long time fixed point state, the length scale $K^{3/2}/\epsilon$ is expected to be constant; thus,

$$\frac{\dot{\epsilon}}{\epsilon} = \frac{3}{2}\frac{\dot{K}}{K} \tag{33}$$

and therefore $C_{\epsilon 1} = C_{\epsilon 2} = 3/2$. This value for $C_{\epsilon 1}$ is found by Dejoan and Schiestel [15] and is consistent with a constant length scale (cf [16]). To summarize, the model ϵ equation

$$\dot{\epsilon} = \frac{3}{2}\frac{\epsilon}{K}(P - \epsilon) \tag{34}$$

is consistent with both the short and the long time limits of ramp flow and is the unique model having this property. The two-equation model for ramp flow consists of the energy equation (4) and the ϵ equation (34).

6. Evaluation of the two-equation model

The evolution of K and ϵ predicted by the two-equation model is compared with the predictions of the CMSB closure in figure 1. The excellent agreement for K and ϵ at long times, in which both quantities exhibit power-law growth, corroborates the formulation of equation (34). However, although K is also well predicted during the transient evolution, the evolution of ϵ reveals significant discrepancies: the two-equation model predicts a much earlier onset of growth of ϵ than the closure, and when ϵ does begin to grow, the closure predicts that it grows much more rapidly.

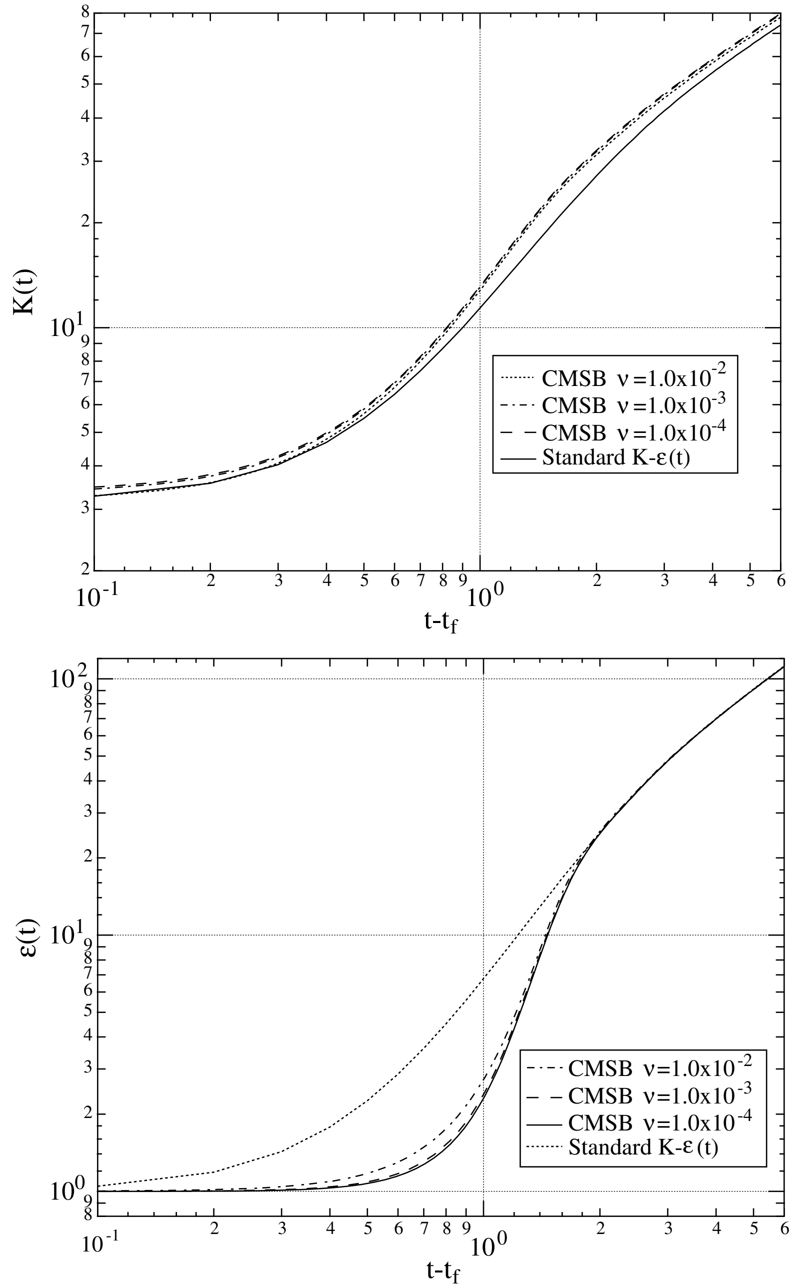

Figure 1. Evolution of kinetic energy K (upper panel) and dissipation rate ϵ (lower panel): CMSB model at different values of Reynolds number compared with predictions of the two-equation model.

These discrepancies reflect an unavoidable defect of the two-equation model: in the two-equation model, ϵ is produced by P, whereas in the exact result, equation (21), it is produced by S. There are two consequences: (i) since equation (25) suggests that S responds to relatively small scales, but the growth of P immediately energizes the largest scales of motion, ϵ inevitably grows too soon in the two-equation model. However (ii), once S begins to grow, the dynamics of S and G is determined by the fast time scales of relatively small scales of motion. The dynamic relevance of these scales is demonstrated by the rapid rate of growth of ϵ in the closure. Such time scales are necessarily absent from the two-equation model.

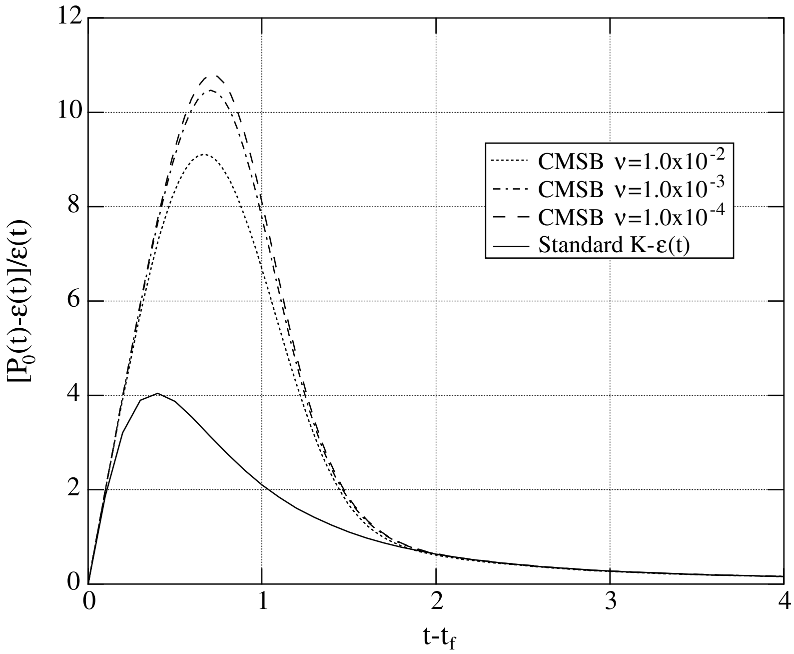

Figure 2. The ratio of production to dissipation in the two-equation model and in the CMSB closure at different Reynolds numbers.

The picture is entirely different at long times, when the growth of ϵ is determined by the $O(Re^0)$ imbalance between S and G: because this imbalance is determined by Yoshizawa's spectral correction, which is dominated by large scales, the growth of ϵ is governed by the integral time scale and the two-equation model becomes valid. We also note that in the asymptotic state, the problem becomes scale-invariant so that only one time scale exists: the proportionality

$$P_2 + S - G \propto \frac{\epsilon}{K}(P - \epsilon) \tag{35}$$

is then almost a geometric necessity; we refer to [6, 9] for a discussion of the fundamental group-theoretic issues.

The proposed relevance of rapid time scales in the transient evolution of the dissipation rate is closely related to the suggestion of Speziale and Bernard [8] that vortex stretching might *not* be balanced by enstrophy destruction. In the context of the dissipation rate equation, this *unbalanced vortex stretching* is modelled by replacing the turbulent time scale K/ϵ by the Kolmogorov time scale $\sqrt{\nu/\epsilon}$. The consequent explosive growth of ϵ leads to a balance of production and dissipation in homogeneous shear flow and to fully self-similar t^{-1} decay of homogeneous isotropic turbulence.

To assess the possible relevance of unbalanced vortex stretching in transient ramp flow, we calculated the ramp flow at different Reynolds numbers. The Reynolds number effects are noticeable but not very strong. Nevertheless, we believe that the rapid growth of ϵ shows the essential relevance of the Speziale and Bernard ideas: the dynamics of the dissipation rate can be influenced by the rapid time-scale characteristics of small scales of motion; however, this relevance appears to be limited to the transient regime. At long times, the standard picture of a balance between vortex stretching and enstrophy destruction is recovered.

Another way to look at the dynamics of ϵ is through the ratio P/ϵ. Figure 2 compares this ratio as computed by the two-equation model and the CMSB closure at three different Reynolds numbers. The graphs again reveal some Reynolds number dependence for the closure model,

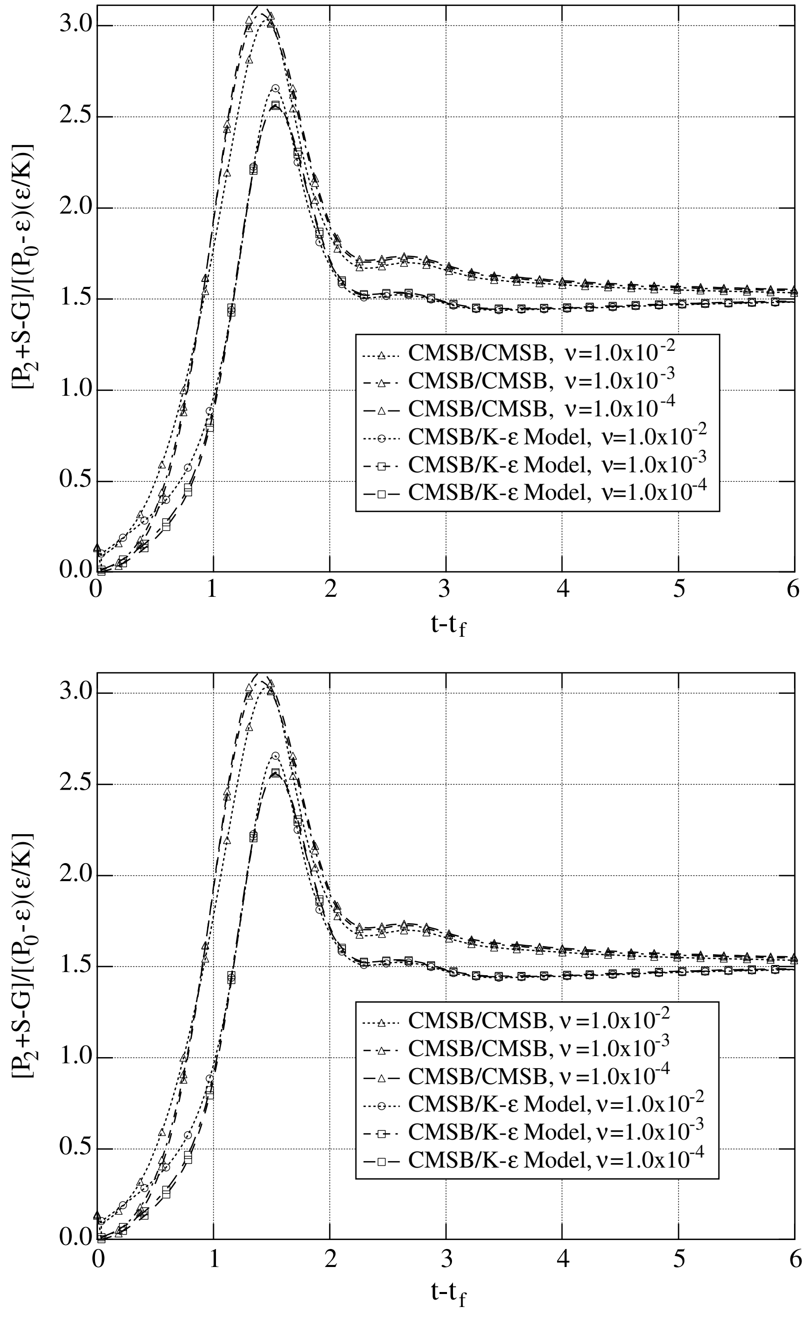

Figure 3. Ratio of the exact value of $\dot{\epsilon}$ to the value predicted by the two-equation model with values of K and ϵ taken both from the CMSB closure and the two-equation model. Results corresponding to two different initial conditions are compared (see text).

which is necessarily absent from the two-equation model but, more significantly, shows that this ratio peaks too early at too small a value in the two-equation model.

The central difficulty of two-equation modelling can be clarified by direct comparison of the right-hand sides of equations (21) and (34). This comparison is shown in the upper panel in figure 3, with K and ϵ computed both from the two-equation model and from the closure: note however that in this comparison, the factor 3/2 on the right-hand side of equation (34)

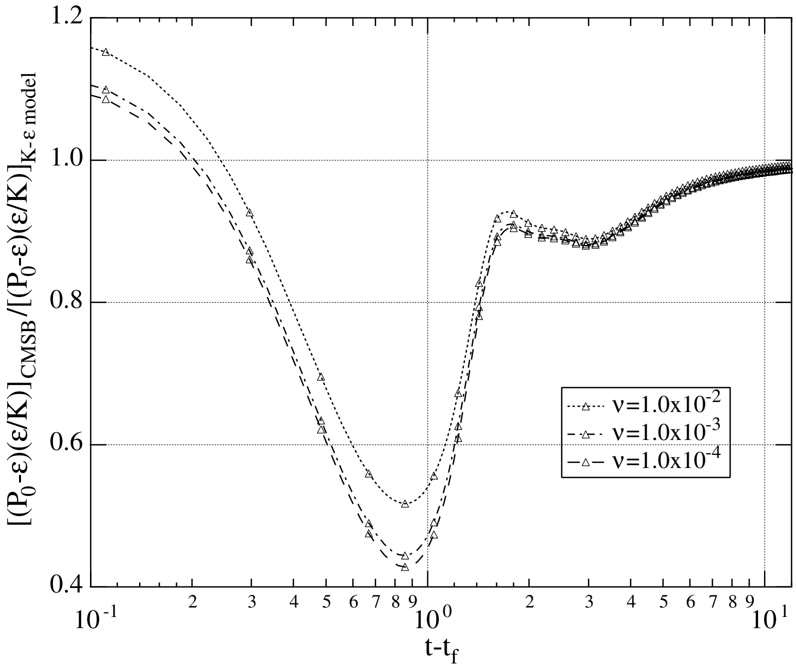

Figure 4. Ratio of right-hand side of the ϵ equation computed with values of K and ϵ from the two-equation model and from the CMSB closure.

has been suppressed. The ratio, which is nearly constant and very close to 1.5 at long times, indicates that equation (34) is indeed a very good model for the asymptotic state; however, it is not satisfactory during the transient.

The effects of Reynolds number in these problems appear to depend on the initial conditions. The lower panel in figure 3 shows results for a much smaller initial value of the time-scale ratio $K/(\epsilon\tau)$: in this graph, K and ϵ are taken from the CMSB model only. The Reynolds number dependence is clearly greatly enhanced by this increase in the dimensionless ramp rate.

A comparison which is implicit in figure 3 is the difference between the right-hand side of equation (34) computed from the two-equation model and from the closure. This comparison is shown in figure 4 for initial conditions corresponding to the upper panel in figure 3. These quantities become equal at long times, but are significantly different during the transient evolution.

7. Comparison with DNS

We briefly discuss the reliability of simplified closures for this problem. Closures in general have been criticized [17] as intrinsically 'too non-linear'. This criticism could certainly bear on the present results, but there are also contrary arguments [18].

In comparison with a complete Lagrangian closure [19, 20], the CMSB model is partially Markovianized and only considers restricted triad interactions. As a two-state variable (E and θ) Markovian model, the CMSB model is similar to the three-state variable test-field model [21], but is unlike EDQNM which is Markovian in E alone. Even partial Markovianization can have an important effect on the short time dynamics [18]; however, it is likely that any degree of Markovianization will tend to accelerate vortex stretching, not retard it. These effects of time dependence will be assessed by computations now in progress with multiple-state variable Markovian closure models.

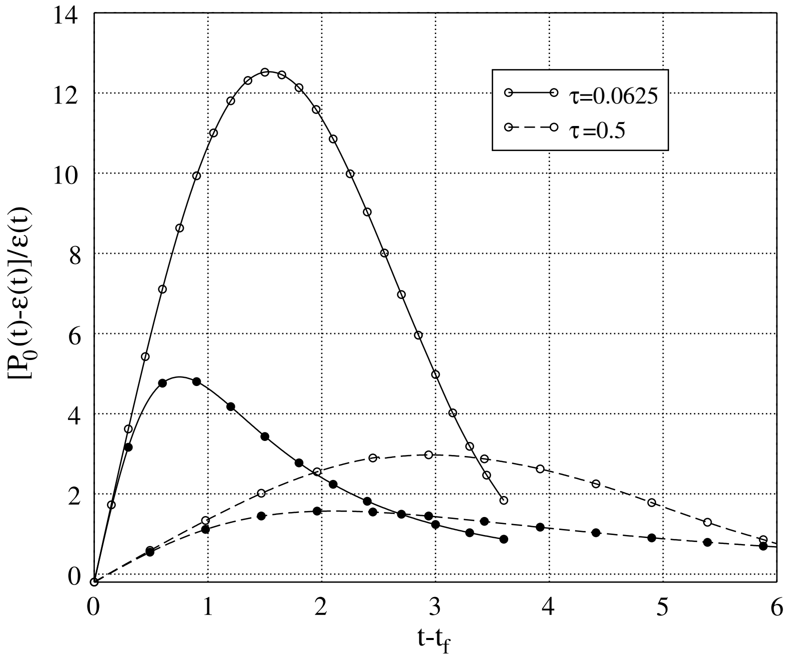

Figure 5. The ratio of production to dissipation in DNS (○) and in the two-equation model (●) at different ramp rates.

The limitation to a restricted class of triad interactions is potentially very significant: because it treats energy transfer through a one-dimensional model, the CMSB model is perhaps more closely related to shell models of turbulence than to the Navier–Stokes equations themselves. Indeed, one might speculate that some form of the CMSB model is in fact the one-loop closure for a suitable shell model. In shell models, the effective restriction of the non-linear interactions typically leads to 'noisier' simulations than comparable DNS. This limitation suggests that the analogous increased coherence of the CMSB model will overstate effects like unbalanced vortex stretching. To address these questions, we will show some preliminary evidence from DNS supporting our main conclusions.

The Navier–Stokes equations were solved in a box containing 128^3 grid points using a standard pseudo-spectral algorithm [22], fully de-aliased by a combination of truncation and phase shifting. The turbulence is sustained by a deterministic forcing term [23, 24] $f_i(\boldsymbol{k}) = A\hat{u}(\boldsymbol{k})/[2E(k)]$ for $0 < k < 1.5$ and $f_i(\boldsymbol{k}) = 0$, otherwise. This yields $\Pi(1) = A$ and $\Pi(k) = 0$ for $k > 1$. For all cases $\nu = 0.004$. The equations are integrated with $A = 0.06$ until the flow reaches stationarity, at which time $Re_\lambda = 90$; then the linear increase in the forcing magnitude given by equation (3) is imposed. Two different ramp rates were chosen: $\tau = 0.5$ and 0.0625. The computations are stopped before ηk_{max} becomes <1.0, where k_{max} is the maximum resolved wavenumber and η is the Kolmogorov microscale. All the results presented represent averages over 10 realizations.

Figure 5 compares the ratio $(P - \epsilon)/\epsilon$ obtained from DNS and the two-equation model. In agreement with the closure results shown in figure 2, figure 5 confirms that the two-equation model predicts that this ratio peaks too early and at too small a value. Later, the predictions of the two-equation model and the DNS become closer. However, the simulations were stopped before reaching the asymptotic state and the performance of the model could not be verified at long times. It should be noted that the comparisons between closure and DNS are qualitative at this point because of differences in the simulation parameters.

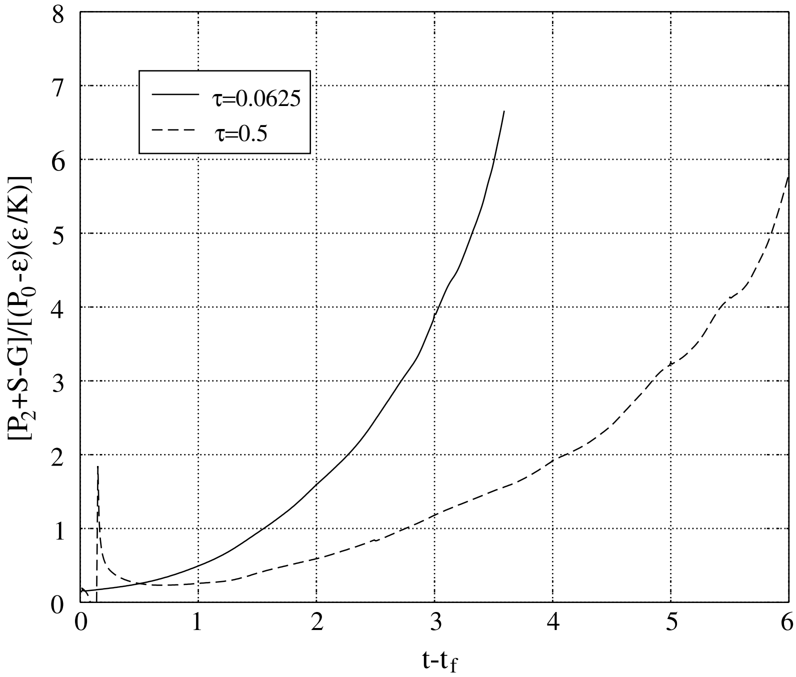

Figure 6. Ratio of the actual rate of change of dissipation rate to the right-hand side of the ϵ model equation computed with values from DNS.

Figure 6 presents results analogous to those of figure 3: it shows the ratio of the value of $\dot{\epsilon}$ computed from DNS to the right-hand side of the ϵ equation (34), with both K and ϵ evaluated from the DNS. At the times that the calculations were terminated, this ratio is continuing to increase rapidly, indicating that ϵ is growing much more rapidly than the right-hand side of equation (34) predicts, even when it is evaluated using the correct values of K and ϵ.

Figure 7 compares the right-hand side of equation (34) evaluated using K and ϵ from the DNS and from the two-equation model; this figure can be compared with figure 4 in which the same comparison is made between the closure and the two-equation model. Using these data to evaluate the ratio of $\dot{\epsilon}$ from the DNS to the prediction of the two-equation model shows that at time 2, the ratio is about 0.4 for the higher ramp rate ($\tau = 0.0625$) and about 0.3 for the lower ($\tau = 0.5$); when the calculations terminate, this ratio is approximately 3.0 for both ramp rates. These values, therefore, confirm the trend noted in section 6: initially, ϵ grows considerably more slowly than the two-equation model predicts, but once growth sets in, it is far more rapid than the growth predicted by the two-equation model.

Clearly, the ratio plotted in figure 6 has not yet peaked in either case, much less has it achieved a constant asymptotic value. This observation underscores the great difficulty of DNS in this problem: once the effect of the ramp is felt at all scales, the rapid growth of small-scale energy begins to destabilize the numerics, requiring ever smaller time steps and, as the Kolmogorov scale begins to decrease, ever more grid points. Although it is very difficult to track the rapid and explosive changes in transient ramp flow with DNS, further DNS studies of ramp flow are planned to validate the more detailed predictions of the closure model.

8. Conclusions

The long time analysis of ramp flow shows the $k^{-7/3}$ corrections to the energy spectrum due to non-equilibrium effects predicted by Yoshizawa [4]. The analysis also corroborates the

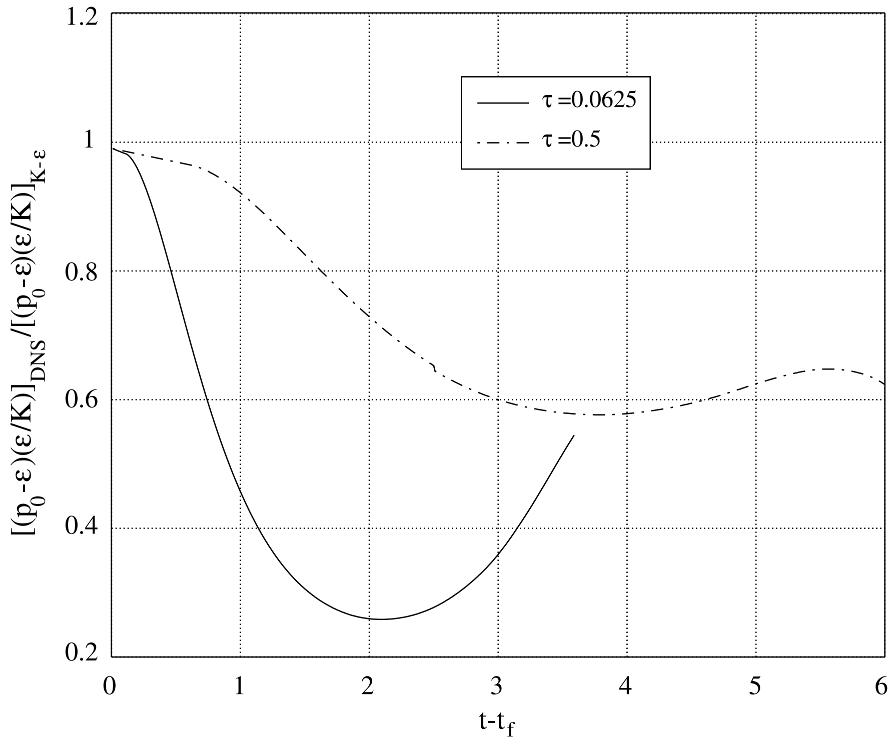

Figure 7. Ratio of the right-hand side of the ϵ equation computed with values of K and ϵ computed from the two-equation model and from DNS.

assumption of an unsteady balance of $\mathrm{O}(Re^{1/2})$ terms made in formulating the ϵ equation. A two-equation model is consistent with the time dependence of the kinetic energy and dissipation rate at both short and long times. However, the transient regime reveals the relevance of fast time scales of small scales of motion which can be compared with the unbalanced vortex stretching discussed by Speziale and Bernard [8, 12].

It is widely accepted that the ϵ equation lacks fundamental justification; it is therefore not surprising that it can fail in very special circumstances like transient ramp flow. At the same time, following previous work at Los Alamos National Laboratory [6, 7], we find theoretical support for the ϵ equation in an asymptotic regime in which self-similarity imposes strong geometric constraints on the time and length scales. Thus, this paper attempts to advance the understanding of both the conditions under which the ϵ equation is valid and of its theoretical limitations.

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