LACUNARITY IN A BEST ESTIMATOR OF FRACTAL DIMENSION

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A best estimator derived by Takens for estimating the dimension of a strange attractor from a discrete set of points is shown to be sensitive to lacunarity in the fractal set.

1. Introduction

Though computing the dimension of fractal sets has achieved widespread popularity (the literature is vast; the three reviews mentioned in ref. [1] are by no means exhaustive), it has proven to be surprisingly difficult to characterize the accuracy and reliability of these computational estimates [2,3] ("It is not difficult to develop an algorithm that will yield numbers that can be called dimension, but it is far more difficult to be confident that those numbers truly represent the dynamics of the system" [2]). It is, perhaps, not surprising that a geometrical object that is called a "strange attractor" should prove to be ill-behaved and hard to analyze. In this Letter, we will address a particular source of systematic error, due to "intrinsic oscillations" or "lacunarity" [4–8], which in particular affect an otherwise very efficient estimator of dimension derived by Takens [9].

2. Dimension algorithms

A variety of algorithms has been proposed for estimating the fractal dimension of a geometrical object (such as a strange attractor) from a discrete sample of points on the set. For an incomplete survey, see refs. [10–16]. In each case, a scaling of something like "mass" as a function of something like "size" yields the dimension. In this Letter, we will concentrate on the pointwise dimension [11] and the correlation dimension [15], though some of what is said applies to the other algorithms as well.

2.1. Invariant measure

As algorithms, the input is necessarily a discrete set of points which sample the set. For the purposes of this Letter, we will consider the limit of $N \to \infty$ points. We will assume that these points fill the set ergodically, and so allow us to define an invariant measure μ. If $\mathscr{A}$ is the attractor, and $S \subset \mathscr{A}$, then the measure of S is defined to be the fraction of points that are in S,

$$\mu(S) = \lim_{N\to\infty} \frac{\#\{x_i \mid x_i \in S\}}{N} . \tag{2.1}$$

Here "#" denotes the cardinality of the set.

2.2. Pointwise dimension

To define the pointwise dimension at the point $x \in \mathscr{A}$, consider the measure of the ball $\mathscr{B}_x(r)$ which has radius r and is centered at the point x. Call that measure $B_x(r) = \mu(\mathscr{B}_x(r))$. Empirically, we estimate $B_{x_i}(r)$ by counting how many neighbors there are within r or x_i:

$$B_{x_i}(r) = \lim_{N\to\infty} \frac{\#\{x_j \mid j \neq i \text{ and } \|x_j - x_i\| < r\}}{N-1} . \tag{2.2}$$

If we have $B_x(r) \sim r^{d_p}$, then d_p is the pointwise dimension at x. Formally,

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$$d_{\mathrm{p}}(x)=\lim_{r\to 0}\frac{\log B_{x_i}(r)}{\log r}\,. \tag{2.3}$$

2.3. Correlation dimension

An effective and very popular algorithm was developed by Grassberger and Procaccia [15] and Takens [16], and is based on the distances r_{ij} between the points x_i and x_j. A correlation integral is defined by the fraction of distances less than a fixed distance r,

$$C(N,r)=\frac{\#\{r_{ij}\,|\,i<j\leqslant N \text{ and } r_{ij}=\|x_i-x_j\|<r\}}{\#\{r_{ij}\,|\,i<j\leqslant N\}}\,. \tag{2.4}$$

Equivalently,

$$C(N,r)=\frac{1}{N}\sum_{i=1}^{N} B_{x_i}(r)\,, \tag{2.5}$$

or

$$C(N,r)=\frac{1}{N(N-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{N} H(r-r_{ij})\,, \tag{2.6}$$

where $H(x)$, the Heaviside function, is zero if $x<0$ and one if $x\geqslant 0$. In terms of the invariant measure, we can write

$$C(r)=\lim_{N\to\infty} C(N,r)$$
$$=\int_x\int_y H(r-\|x-y\|)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\,. \tag{2.7}$$

The correlation dimension ν expresses the scaling $C(r)\sim r^{\nu}$; or more formally,

$$\nu=\lim_{r\to 0}\frac{\log C(r)}{\log r}\,. \tag{2.8}$$

We have used the ambiguous symbol "$\sim$" to denote the (equally ambiguous) phrase "tends to be approximately proportional to". We, in particular, have avoided saying $C(r)\propto r^{\nu}$, since this proportionality may not hold, even as $r\to 0$. Indeed, the failure of this proportionality is the essential feature of lacunarity. We can certainly write

$$C(r)=\Phi(r)r^{\nu}\,, \tag{2.9}$$

which *defines* $\Phi(r)$, since ν is already defined in eq. (2.8). *It is the behavior of this prefactor* $\Phi(r)$ *which is our concern in this Letter.* We tend to think of $\Phi(r)$ as a coefficient which approaches a constant as $r\to 0$, but all that eq. (2.8) requires is that

$$\lim_{r\to 0}\frac{\log\Phi(r)}{\log r}=0\,. \tag{2.10}$$

Further, since $C(r)$ is necessarily nonzero, we also have that $\Phi(r)$ is always nonzero.

In practice [#1], we cannot take $r>0$, so we settle for some approximation at finite $R>0$. One obvious choice is

$$\nu(R)=\frac{\log C(R)}{\log R}\,. \tag{2.11}$$

This estimate does have the desirable property that $\nu(R)$ converges to the correct dimension ν as $R\to 0$. However, if $\Phi(r)$ is equal to a constant, then

$$\nu(R)=\frac{\log C(R)}{\log R}=\frac{\log(\Phi R^{\nu})}{\log R}=\nu+\frac{\log\Phi}{\log R} \tag{2.12}$$

approaches ν with logarithmic slowness as $R\to 0$.

One alternative is to take

$$\nu(R_1,R_2)=\frac{\log C(R)-\log C(R_2)}{\log R_1-\log R_2}\,, \tag{2.13}$$

which does avoid the logarithmic slowness described above, though it involves the choice of *two* scales, R_1 and R_2, both of which should tend to zero. It is more common for estimates of dimension to be taken by fitting a slope to a log–log plot of $C(r)$ versus r over some range $r<R$, as was originally advocated by Grassberger and Pocaccia [15]. This again avoids the logarithmic slowness, but there is no clear criterion for finding a best fit. Unweighted least squares is a particularly poor choice for two reasons. One, an unweighted fit effectively assumes a uniform error in $\log C(r)$. This is wrong because the statistics are much poorer at small r than at large r. Two, even a weighted fit assumes that errors in $\log C(r)$ are independent of each other. But $C(r+\Delta r)$ is equal to $C(r)$ plus the fraction of distances between r and $r+\Delta r$. Paticularly, for small Δr, the error at

[#1] In practice, the real limitation is finite N, but an indirect effect of this is that we must limit r to some positive value.

$\log C(r+\Delta r)$ will be strongly correlated with the error at $\log C(r)$.

A more sophisticated idea is to use a maximum likelihood estimator. (This idea was introduced by Fisher [17]; a recent exposition can be found in ref. [18].) The idea is to *assume* that distances r_{ij} are distributed randomly according to a probability

$$P(r_{ij}<r)=C(r)=\Phi r^{\nu} . \tag{2.14}$$

Then, calculate the likelihood of observing the particular distances r_{ij} that *are* seen, given this assumption about their probability distribution. This likelyhood is expressed as a function of the parameters ν and Φ; the values of ν and Φ which maximize the function comprise the maximum-likelihood estimator.

This approach was applied by Takens [9] to the correlation dimension. Assuming the form of probability distribution above, Takens derives an optimal estimator

$$\nu(R)=-\frac{1}{\langle\log(r_{ij}/R)\rangle} , \tag{2.15}$$

where the angle brackets $\langle\ \rangle$ denote an average over all distances r_{ij} which are less than R. In terms of the correlation integral, this can be written [3]

$$\nu(R)=\frac{C(R)}{\int_0^R [C(r)/r]\,\mathrm{d}r} . \tag{2.16}$$

We note that an identical prescription can be applied to the pointwise dimension. Here, the estimator for the dimension at point x is

$$d(x,R)=\frac{B_x(R)}{\int_0^R [B_x(r)/r]\,\mathrm{d}r} . \tag{2.17}$$

As we will see, this is a more efficient estimator of dimension than the straight ratio of logarithms given in eq. (2.11), in the sense that $\nu(R)$ approaches the dimension ν much more quickly as $R\to 0$. However, it is the purpose of this Letter is to probe that for fractal sets with lacunarity, $\nu(R)$ does not converge at all. In fact, Takens observed numerically that estimates of dimension for the Zaslavskii map [19] fluctuated, depending on the choice of R. This fluctuation was correctly attributed to the failure of the assumption that Φ was constant. For Φ constant, or if it approaches a constant as $R\to 0$, the Takens estimator behaves correctly. In fact, we will derive a condition on the behavior of $\Phi(r)$ which is more strict than eq. (2.10), and under which the Takens estimator converges.

We are compelled nonetheless to comment that for purposes of practical approximation, the Takens estimator may still be a very useful formula.

3. Convergence criterion for the Takens estimator

In this section, we derive a condition on the prefactor $\Phi(r)$ which allows the Takens estimator to converge. First write

$$\nu(R)=\frac{C(R)}{\int_0^R [C(r)/r]\,\mathrm{d}r}=\frac{\Phi(R)R^{\nu}}{\int_0^R \Phi(r)r^{\nu-1}\,\mathrm{d}r} . \tag{3.1}$$

Now, take $z=-\log(r/R)$, so $r=R\mathrm{e}^{-z}$, and $r\to 0$ becomes $z\to\infty$:

$$\begin{aligned}\nu(R)&=\frac{\Phi(R)R^{\nu}}{\int_0^\infty \Phi(R\mathrm{e}^{-z})(R\mathrm{e}^{-z})^{\nu-1}(-R\mathrm{e}^{-z}\,\mathrm{d}z)}\\ &=\frac{\Phi(R)}{\int_0^\infty \mathrm{e}^{-\nu z}\Phi(R\mathrm{e}^{-z})\,\mathrm{d}z} .\end{aligned} \tag{3.2}$$

This is an approximation to ν with relative error

$$E(R)=\frac{\nu(R)-\nu}{\cancel{\nu(R)}} . \tag{3.3}$$

Invoking the identity $\int_0^\infty \mathrm{e}^{-\nu z}\mathrm{d}=1/\nu$, we can write

$$E(R)=\frac{\cancel{\int_0^\infty \mathrm{e}^{-\nu z}[\Phi(R\mathrm{e}^{-z})-\Phi(R)]\,\mathrm{d}r}}{\Phi(R)} . \tag{3.4}$$

In general, we have that the Takens estimator converges iff

$$\lim_{R\to 0}\int_0^\infty \mathrm{e}^{-\nu}[\Phi(R\mathrm{e}^{-z})/\Phi(R)-1]\,\mathrm{d}z=0 . \tag{3.5}$$

We first comment that for a nonlacunar fractal, with Φ constant, we have $\Phi(R\mathrm{e}^{-z})=\Phi(R)$, the error $E(R)$ is zero, and the estimator is accurate. Further, for a fractal in which Φ approaches a constant, the error vanishes as $R\to 0$.

We emphasize that this condition is not equivalent to eq. (2.10). It is possible to devise functions $\Phi(r)$ for which eq. (2.10) is satisfied, but eq. (3.5) is not.

Such functions, we will see, arise from lacunar fractals.

4. General lacunarity

Mandelbrot [4] introduced the term lacunarity as a measure of the "texture" of a fractal. There is no generally accepted definition of lacunarity, and indeed there has been some controversy over how it should be measured (a brief account, offered as an aside, can be found in ref. [20]). We follow ref. [7], among others, in using the term to refer to the variability of $\Phi(r)$, particularly where $\lim_{r\to 0}\Phi(r)$ is not a positive constant. In such cases, the Takens estimator usually fails to converge.

4.1. Periodic lacunarity

Fig. 1 shows a log–log plot of the correlation integral for the middle-thirds Cantor set. (This example is also discussed in ref. [7].) Also plotted is $\Phi(r)=C(r)/r^{\nu}$. We see that $\Phi(r)$ has the property

$$\Phi(r)=\Phi(Lr)\ , \tag{4.1}$$

for the constant $L=1/3$ and for all r. We take eq.

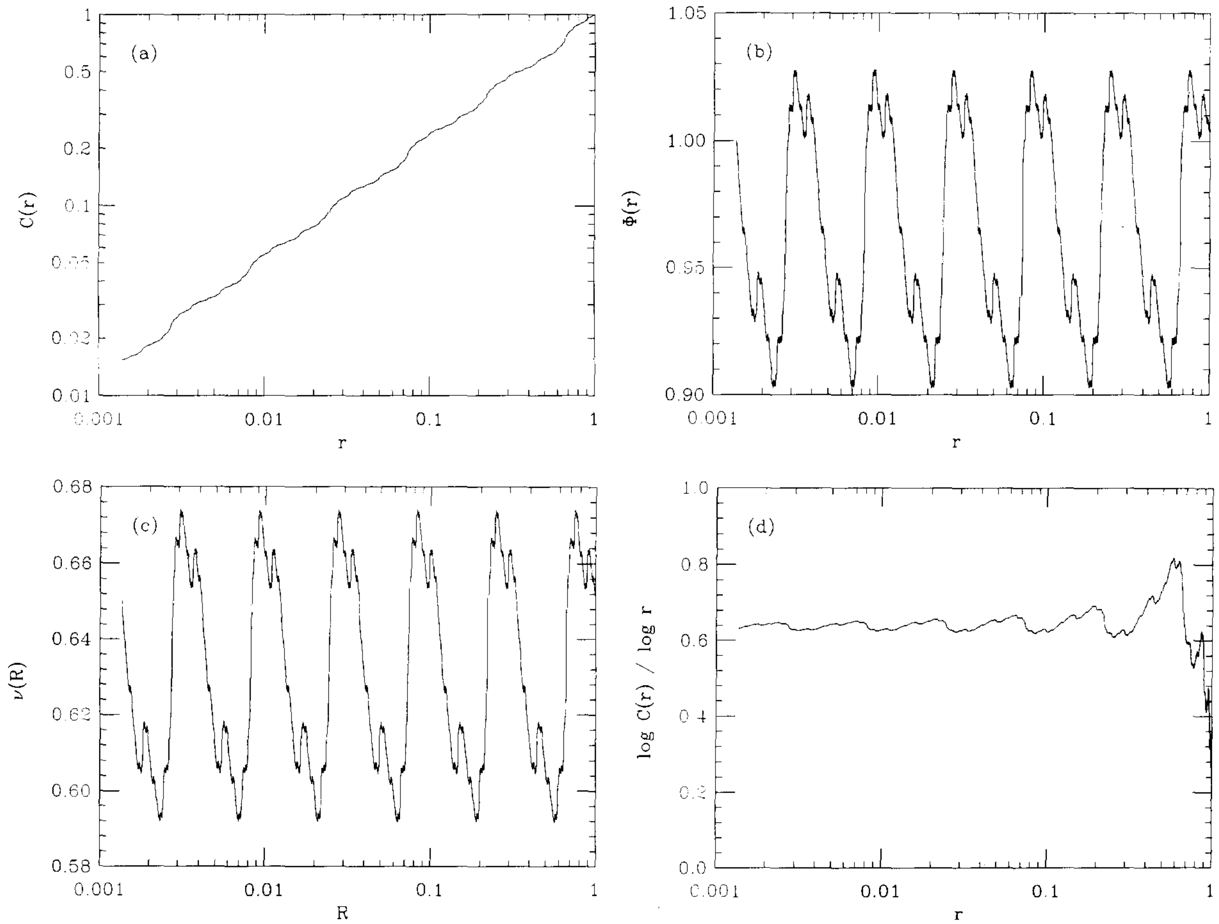

Fig. 1 The middle-thirds Cantor set. (a) Log–log plot of the correlation integral $C(r)$. The overall slope is $\nu=\log 2/\log 3\approx 0.63$, which is the dimension of the fractal set. (b) The periodic lacunar function $\Phi(r)$. Hence $\Phi(r)=\Phi(r/3)$ for all r. (c) The Takens estimator $\nu(R)$ does not approach the correct dimension ν, but instead oscillates without damping as $R\to 0$. (d) A more reliable estimator is the ratio of logarithms, $\log C(R)/\log R$. Although we point out in the text that the convergence of this estimator is usually very slow, approaching the limit as $O(1/\log R)$; in this particular example, the coefficient of that leading term is zero.

(4.1) as a definition for periodic lacunarity. As a function of $z=-\log(r/R)$, we see that $\Phi(Re^{-z})$ is periodic with "wavelength" $\log L$. Further, we see immediately from eq. (3.2) that periodic lacunarity implies

$$\nu(R)=\nu(LR)\,. \tag{4.2}$$

It follows that $\nu(R)=\nu(L^n R)$ for arbitrarily large n. In particular, this means that if $\nu(R)\neq\nu$ for any $R>0$, then $\lim_{R\to 0}\nu(R)$ cannot be ν. Thus, we need only show a single value of R for which $\nu(R)\neq\nu$ to prove nonconvergence of the Takens estimator.

Take R so that $\Phi(R)$ is minimum; there is no difficulty in doing this since $\Phi(R)$ is periodic in the sense of eq. (4.1). In particular, then,

$$\Phi(Re^{-z})/\Phi(R)\geqslant 1\,, \tag{4.3}$$

for all z, with strict inequality for at least a range of values of z (unless Φ is constant!). This implies that

$$\int_0^\infty e^{-\nu z}[\Phi(Re^{-z})/\Phi(R)-1]\,dz\,,$$

is strictly greater than zero. It follows that $\nu(R)\neq\nu$, and from the argument above, that $\lim_{R\to 0}\nu(R)\neq\nu$.

The Takens estimator does not converge for fractals with periodic lacunarity.

4.2. Aperiodic lacunarity

Although cooked-up examples like the middle-thirds Cantor set displays a lacunarity that is precisely periodic as $r\to 0$, a more typical fractal, such as the Hénon attractor [21], exhibits oscillations which are more complicated [#2]. See fig 2. It is claimed in ref. [7] that the amplitude of the oscillations decrease as $r\to 0$ for the Hénon attractor. This is verified numerically by Grassberger [20], who computes dimension according to eq. (2.13), with $R_1=2\epsilon$ and $R_2=\epsilon/2$. He finds that the oscillations that are observed for $\epsilon>10^{-4}$ do indeed damp out for ϵ all the way down to 2×10^{-7}. Turchetti and Vaienti [22] prove that the oscillations are damped for (generic elements of) a class of multi-scale Cantor sets. Nonetheless, as Arneodo et al. [23], point out, these oscillations still make estimation of dimension a more difficult problem.

[#2] For the *pointwise* dimension, nearly precise periodic lacunarity is more common. This periodicity is pointed out in ref. [3] for the pointwise dimension of the Hénon attractor at the fixed point.

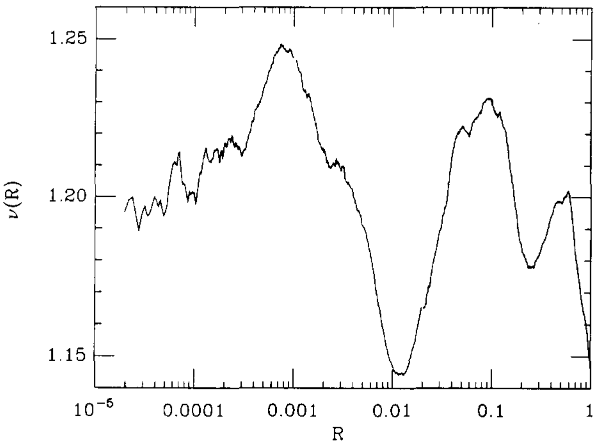

Fig. 2. The Takens estimator $\nu(R)$ for the Hénon attractor. It is expected that $\nu(R)$ approaches a constant as $R\to 0$.

5. Edge effect

In this section, we demonstrate the usefulness of the Takens estimator for a case in which Φ is nonconstant, yet not lacunar. That is, we have $\lim_{r\to 0}\Phi(r)$ exists and is a positive constant. We take as our example the unit interval [0, 1] with uniform density. In this case, ν is clearly one, and the correlation integral is given [3] by

$$C(r)=2r-r^2=(2-r)r\,, \tag{5.1}$$

and

$$\Phi(r)=2-r\,. \tag{5.2}$$

We can write an explicit expression for the Takens estimator, using the formula for $\nu(R)$ in eq. (3.2). We have

$$\nu(R)=\frac{\Phi(R)}{\int_0^\infty e^{-\nu z}\Phi(Re^{-z})\,dz}$$

$$=\frac{2-R}{\int_0^\infty e^{-z}(2-Re^{-z})\,dz}$$

$$=\frac{2-R}{2-R\int_0^\infty e^{-2z}dz}=\frac{2-R}{2-R/2}\approx 1-R/4\,. \qquad (5.3)$$

We see that the error scales as $O(R)$ for the Takens estimator; this is significantly faster than the $O(1/\log R)$ scaling that is achieved by the more "reliable" estimator in eq. (2.11).

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