The Mechanism of Strong Nonlinear Elasticity in Earth Solids

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Abstract. This work describes several physical models of the mechanical properties of rocks resulting in macroscopic nonlinear behavior due to their mesoscopic-scale structure. Theoretical models include Hertzian grain contacts with multiple scales and hysteretic properties. A significant addition to the highly nonlinear response in these materials are fluids contained in the soft bond system.

INTRODUCTION

The nonlinear response of earth materials is an extremely rich topic, one that has broad implications in earthquake engineering, nondestructive testing and material science (e. g., [7]). The mechanical properties of rocks appear to be a part of a broader class, one we call the *Nonlinear Mesoscopic Elasticity*, or *Structural Nonlinear Elasticity* class. Mesoscopic materials can be thought of as composed of a hard matrix material (grains, crystals) containing soft features (the bond system, cracks) that lead to a very large nonlinear response, including hysteresis and relaxation (slow dynamics). It is also becoming clear that water in the bond system contributes significantly to the nonlinear response in these materials; however, exactly how the bond system and fluids contribute to the nonlinear response is not yet well understood. Here we discuss possible mechanisms of nonlinearity in rock based on observations.

Acoustic nonlinearity may manifest itself in a variety of manners, including nonlinear attenuation, resonant frequency shift, harmonic and subharmonic generation, and slow dynamics. Most quantitative measurements have been performed using resonant bar experiments. Due to the resonance amplification, it is perhaps the most sensitive manner by which to observe nonlinear behavior, even at dynamic strains as small as $\varepsilon = 10^{-9}$ (in one dimension, $\varepsilon = \partial u/\partial x$ where u is the displacement). The quantitative indicators of nonlinear behavior from dynamic (vibroacoustic) experiments are the amplitudes of wave harmonics, amplitudes of wave cross-modulation, resonance frequency shift, amplitude-

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dependent loss, and their respective scalings with strain amplitude. Here we focus on developing models that may predict scaling relations derived from resonance experiments.

Examples of these results are numerous. The data illustrate that, first, the nonlinear responses in rocks are large and second, that the nonlinear dependencies can be different from those of classical, atomic elastic media. Thus, the frequency shift is often proportional to the strain amplitude ε_0 rather than to ε_0^2 as for the classical "cubic" nonlinearity, and the third harmonic amplitude is proportional to ε_0^2, not ε_0^3. This implies that the stress-strain dependence $\sigma(\varepsilon)$ in rocks cannot ordinarily be described by a standard Taylor series expansion of the stress-strain relation. In these cases, nonclassical behavior must be appealed to.

THEORETICAL MODELS FOR STRONG ELASTIC NONLINEARITY

The theoretical models of medium nonlinearity from the traditional elasticity theory are based on application of expansions of the strain energy function or stress-strain equation of state keeping low-order (quadratic and/or cubic) nonlinear terms. From the energy expansion in the classical case, a similar series for stress-strain relation can be written:

$$\sigma = M\left(\varepsilon + \beta\varepsilon^2 + \delta\varepsilon^3 + \ldots\right), \tag{1}$$

where M is the elastic modulus, and β and δ are nonlinear coefficients that can be expressed in terms of Landau or Murnaghan moduli. Although this relation has been observed in some experiments with rocks, it is relatively rare. More often than not, the nonclassical behavior mentioned above is observed. This behavior cannot be described by the above equation. Further, the values of nonlinear coefficients are typically orders of magnitude larger than those due to atomic nonlinearity. To explain these facts, one must appeal to more general models of static and dynamic behavior of rock, which include non-analytical stress-strain dependencies.

Hertzian Contacts

A relevant starting model of nonlinearity in rock is based on representing the rock as a system of dry, contacting grains (e. g., [2]) (Fig. 1).

These contacts are much softer than the matrix material, and therefore play the primary role in nonlinear elastic response of the medium. In this model, the distance change Δ between the grain centers is related to the compressive force, F_h, by the Hertzian contact law [3],

$$\Delta = \left(\frac{3(1-\nu^2)F_h}{4ER^{1/2}}\right)^{2/3}, \tag{2}$$

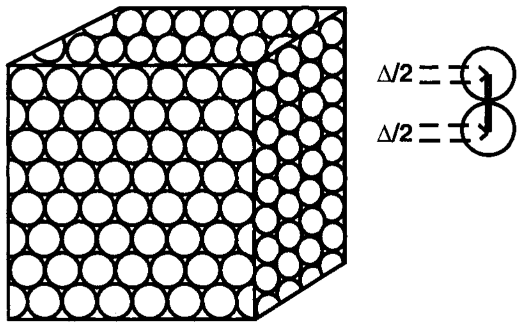

FIGURE 1. An aggregate of contacting grains.

where E is the Young's modulus of the material, ν is Poisson's ratio, and R is the grain radius.

For a dry medium composed of identical spheres this yields the following one-dimensional stress-strain relation

$$\sigma(\varepsilon) = \frac{\overline{n}(1-\alpha)E}{3\pi(1-\nu^2)}\varepsilon^{3/2}, \tag{3}$$

where σ is effective stress, $\overline{n}$ is the average number of contacts per grain and α is the fraction of empty (porous) space per unit volume. For a random packing of unconsolidated grains, $\overline{n} = 8.84$ and $\alpha = 0.392$ (which may not be applicable to rock).

It is evident that the contact contribution to the sound speed, $c = (\rho^{-1}\mathrm{d}\sigma/\mathrm{d}\varepsilon)^{1/2}$, tends to zero at small positive strains (negative strain means that grains just separate, and there are no contact forces at all). However, $\mathrm{d}c/\mathrm{d}\varepsilon$ which is a measure of nonlinearity, goes to infinity. In real experiments the aggregate is subject to a static pressure creating a constant pre-strain ε_0, and for small one-dimensional perturbations, we can expand σ into the series (1), where the modulus is

$$M = \frac{\overline{n}(1-\alpha)E}{2\pi(1-\nu^2)}\varepsilon_0^{1/2}, \tag{4}$$

and the quadratic and cubic nonlinearity coefficients are

$$\beta = 1/2\varepsilon_0; \qquad \delta = 1/6\varepsilon_0^2. \tag{5}$$

Some interesting properties of granular materials follow from the above equation of state: in particular, the nonlinearity parameters (5) do not depend either on grain size or composition, but strictly on the pre-strain, i. e. on static pressure; that implies that the larger the overburden, the smaller is the material nonlinearity which is in line with observation [4].

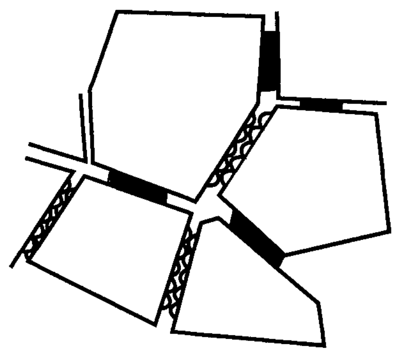

FIGURE 2. Granular medium with two-scale contacts.

Multiscale Contact Structure

The above results were experimentally verified in laboratory experiments using unconsolidated grains. In its simplest version, however, this model is insufficient for rock: it does not include stress-strain hysteretic effects. On the other hand, this model admits many effective extensions. One of them, under present consideration, deals with small-scale multicontact interfaces between grains which yields localized regions of nonlinearity (similar to the "bed of nails" model for a crack considered in [5]). Indeed, there is evidence that the stress path in these materials may not be homogeneous [7]. The following model includes two sizes of contacts. In an attempt to more closely approximate the actual structure, we consider large grains in the form of polyhedrons of a mean size R and surface S, separated by thin, rough contact layers of thickness $2R$ (Fig. 2).

The rough portion of the contact layers are hemispheres of radius $r \ll R$ which take a share s of the surface S, and the remaining space between grains is locked or empty. The volume change of the different structural elements under a given forcing (via the change of elastic energy) is

$$\sigma = \sigma_m + \sigma_c \frac{\delta V_c}{\delta V}, \tag{6}$$

where $\delta V_c / \delta V$ is the relative volume change due to the contacts, and σ_m is the strain due to cemented contacts. For σ_m we have $\sigma_m = M\varepsilon$, where $M = \rho c^2$ (ρ is density and c is the linear sound speed). The last term in Eq. (6) corresponds to the Hertzian relation for the force F_h between two small contacting spheres,

$$F_h = \left(\frac{4E\sqrt{r}}{3(1-\nu^2)} \right) \Delta^{3/2}.$$

The total volume change per grain of volume V_g is $\Delta V = bS\Delta$, where $b = 1$ for bulk compression and $b = 1/3$ for longitudinal strain [2]. The change of contact volume, ΔV_c, is

$sS\Delta$, and its contribution to strain, $\sigma_c = F_h/\pi r^2$. As a result we have

$$\sigma = (\rho c^2)\varepsilon + \left(\frac{F_h}{\pi r^2}\right)\frac{s}{b}. \tag{7}$$

For example, for cubic grains with displacement a (e. g., $\varepsilon = \Delta/a$), $E = 10^{11}$ dyn/cm^2, $\rho = 2$ g/cm^3, and $c = 2 \cdot 10^5$ cm/s, we have

$$\sigma = (\rho c^2)\varepsilon + \left(0.6 \cdot 10^{11}\right)(3s)\left(\frac{R}{r}\right)^{3/2}\varepsilon^{3/2}, \tag{8}$$

not uncharacteristic of some rocks.

Comparison of the nonlinear part of Eq. (8) with Eq. (3) shows that the nonlinearity in the former formula acquires $(R/r)^{3/2}$ which can dramatically enhance the nonlinearity. For the experiments with sandstone, the values $R \simeq 100$ mcm and $r \simeq 0.5$ mcm seem to be realistic and sufficient to explain the experimental resonance frequency shift. In real rock, a more complicated, fractal structure exists, so ultimately this model must be modified [6,7].

The Role of Intergrain Fluids

There exists much evidence that the presence of fluids, particularly at very small degrees of saturation, affects linear properties of rocks (sound speed, dissipation), where it has been suggested that the monolayer and capillary forces may be responsible (e. g., [4]). Recent experiments conducted by B. Zinszner and P. Johnson (unpublished) indicate that the nonlinear response can change significantly with water saturation as well. Figure 3 shows resonance frequency shifts for Meule sandstone, conducted at water saturations ranging from 1-95%. The nonlinear response, as measured by the change in resonance frequency versus strain, shows nearly an order of magnitude increase from 1 to 30% saturation. From 40-95% saturation, the response remains approximately the same. Hence, nonlinear experiments at varying water saturation in sandstone and limestone show even stronger dependencies on saturation than do the linear wave speed and dissipation.

In the following we briefly discuss models of wet Hertzian contacts that contain a physical basis related to the discussion above.

A simplest case is that of a 100% filling of the intergrain space (100% saturation) [2]. In this case, the fluid acts as an additional elasticity to that of the grain contacts. The resulting stress-strain relation (neglecting atomic fluid nonlinearity) has the form,

$$\sigma = \frac{K_f}{\alpha + (1-\alpha)K_f/K_s}\varepsilon + \frac{\overline{n}(1-\alpha)E_s}{3\pi(1-\nu_s^2)}\varepsilon^{3/2}, \tag{9}$$

where K_f and K_s are the bulk modulus of the fluid and solid phase, respectively, and α is the porosity. The above equation illustrates that the effect of pore fluids is to decrease the

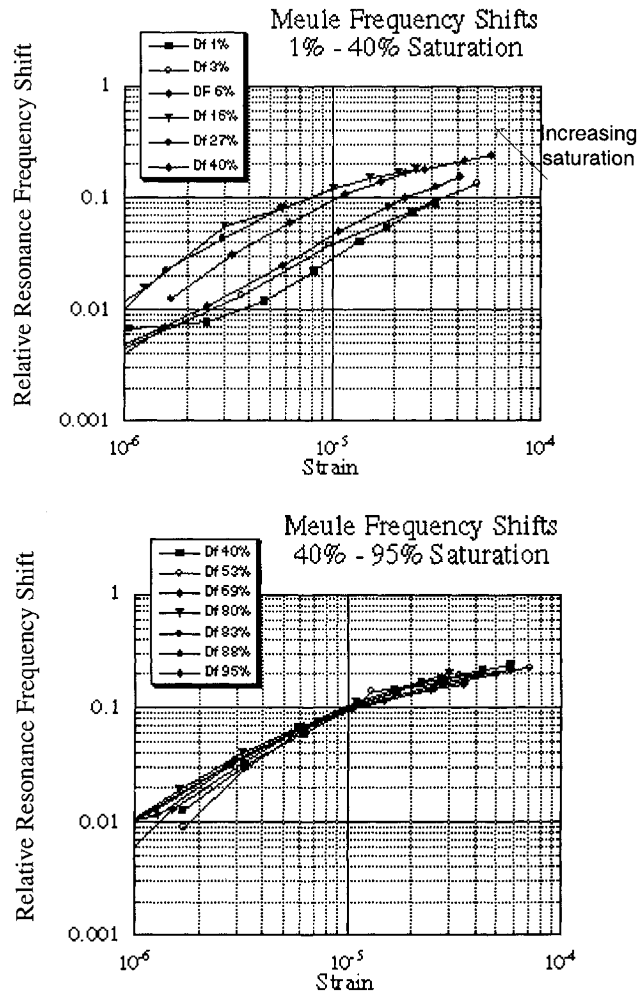

FIGURE 3. Nonlinear response of Meule sandstone under varying saturation conditions.

nonlinearity and increase sound velocity by adding fluid rigidity to that of the solid. The latter effect is, indeed, typically observed for large saturations.

As mentioned, in smaller volume ratios the fluid can, on the contrary, increase the medium nonlinearity. The attempts to describe this effect are associated with capillary and Van-der Waals forces arising at thin contacts between grains. There exists a bulk of literature describing elastic forces arising at thin wet contacts between grains (e. g., [8]).

To estimate the nonlinearity due to fluids, we considered capillary and Van-der Waals forces. In the simple case of a contact between two spheres of equal radii R under precompression where fluid is concentrated between flat surfaces (Fig. 4), the attractive capillary

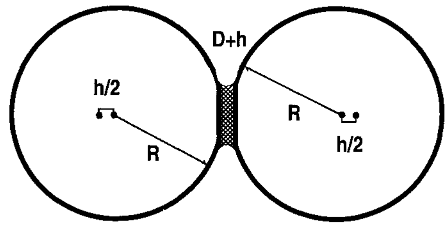

FIGURE 4. Hertzian contacts containing water.

force between them is

$$F_c = \frac{2V\gamma\cos\Theta}{(D+h)^2}, \tag{10}$$

where V is the volume of fluid between grains, γ is the surface tension coefficient, Θ is the angle between surfaces of fluid and solid, D is the equilibrium thickness of the fluid layer, and h is due to external (acoustic) stress.

Considering the change of these parameters due to acoustic deformation at a fixed fluid volume and averaging over the bulk of the material, we find the contribution of capillary forces to the stress-strain relation,

$$\sigma_c \simeq \frac{2V\gamma\cos\Theta}{\pi R^2 D^2(1+2\varepsilon R/D)}. \tag{11}$$

This stress must be added to the Hertzian stress considered above. It is seen from here that: (i) The presence of fluid decreases the linear elastic modulus, $K(\varepsilon) = d\sigma/d\varepsilon$ [hence, the sound velocity – the stronger the larger the saturation (V)]; (ii) K increases for positive strain; and (iii) negative cubic nonlinearity results from the expansion of $K(\varepsilon)$, and, consequently, causes an additional nonlinear shift of resonance frequency in a resonant bar. The first and the third properties agree with the experimental data.

We note that a similar problem was recently considered by Nazarov for cracks. He also discussed viscous motion of fluid (with small Reynolds numbers) [9] and concluded that it can provide much stronger nonlinearity than in the case of an ideal fluid.

The other important case is that of very thin fluid layers between grains, of order a few or a few tens of molecular monolayers (i. e., before capillary condensation takes place). In this case, the dipole Van-der Waals force plays an important role. For a flat contact, this force can be shown to be,

$$F_d = \frac{A}{6\pi(D+h)^3}, \tag{12}$$

where again, D is the thickness of the gap, and A is the Hamaker constant (of order 10^{-13}-10^{-14} erg for water). This effect should also be combined with the Hertzian force.

It should be noted that elasticity of fluid films can be hysteretic (due to different values of Θ for the motions back and forth in Eq. (10) [8]) and also can provide log time relaxation of parameters that is seen in experiment [10]. This is subject of intense research currently.

Models with Hysteresis

Although dynamical hysteresis in metals was observed long ago, for rocks this is a relatively new topic (see [7] and references therein). An adequate equation of state must include the dependence on the history of the process which can be characterized, in the quasistatic case, by the sign of $\dot{\varepsilon}$:

$$\sigma = M\left(\varepsilon + \beta\varepsilon^2 + \delta\varepsilon^3 + \ldots\right) + A\left[\varepsilon, \mathrm{sign}\left(\dot{\varepsilon}\right)\right], \tag{13}$$

where A is a functional describing all "nonclassical" effects. A specific form of A should, strictly speaking, follow from a consideration of the mesoscopic structure of the material. At present, however, only phenomenological representations exist. Some empirical models were suggested for metals, rocks and sand. A more sophisticated and physically suggestive phenomenological model, called the Preisach-Mayergoyz (P-M) space model, that successfully links dynamic and static nonlinearity, and includes hysteretic nonlinear behavior of rock elasticity with discrete memory, was developed in a series of papers by our colleagues (e. g., [11,7]). The P-M space model is based on assuming that the elastic properties of a macroscopic sample of material result from the integral response of a large number of individual elastic elements, some of which are hysteretic. The individual elements are combined for analysis in what is known as P-M space (also referred to as Preisach space). This model has been very successful in describing many static and dynamic features of the rocks noted above (high nonlinearity, hysteresis, discrete memory). However, it is still of a phenomenological nature and as such it lacks a connection with the underlying physics.

A physical model of hysteretic elasticity was suggested by Granato and Lücke (below G-L), as early as in 1956 for metals [12]. They applied an idea developed even earlier, using an analogy between a segment of a dislocation line pinned to impurity particles, and the motion of a string. These dislocations contribute to the volumetric elastic deformation. As the stress increases (they considered primarily shear stress), dislocations deform like pieces of a string until, at some critical stress, they are disconnected from all impurity atoms between the nodes of a crystalline structure. As a result, the material becomes softer, which means a strong volumetric nonlinearity in the stress-strain dependence (Fig. 5). This process is irreversible. Upon reducing stress, the system returns to a relaxed energy state but along a different energy trajectory. Hence, a typical hysteretic loop is formed. It also implies a slow dynamical effect, because the pinnings are restored to energy equilibrium over some time. The distances between the sticking points are statistically distributed, which smooths the hysteretic loop (Fig. 5, dashed line). This model has been developed in many manners (including frequency dependence). In spite of some disadvantages, this was a pioneering micromodel for hysteretic dynamic behavior of elasticity.

We used a model of G-L elements uniformly distributed in their loop sizes in finite limits, found the shear stress, and calculated the integral of the stress-strain loop supposing that the distribution is uniform. The result is

$$\sigma_{NL} = Q\varepsilon\left(\varepsilon_m \pm \varepsilon\right), \tag{14}$$

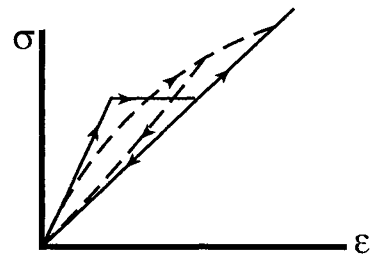

FIGURE 5. Stress-strain dependencies for the Granato-Lücke model.

where ε_m is the strain amplitude, Q is a constant depending on the parameters of the hysteretic elements, and $\pm$ corresponds to the sign of $\partial\varepsilon/\partial t$ (i. e., a branch in the hysteretic loop). This expression is essentially the same as that following from the aforementioned P-M space model for a uniform distribution of rectangular elements. It should be noted, however, that in the G-M model for metals, a non-uniform (exponential) distribution of elements is used leading to a different shape of the hysteretic loop, albeit with similar qualitative characteristics.

A variation of the G-L model was considered by Asano [13]. It permits a contact to slide within some limits (a "slider") rather that to be detached as in the G-L model (a "ratchet"). The corresponding hysteretic loop will surround the zero stress-strain point rather than including it. This model can also give a dependence of the type (14). This study is in progress.

For rock, the role of dislocations is not clear. Certainly in amorphous bond material such as silica (typical of many sedimentary rocks), dislocations cannot be appealed to. In solids, however, where crystalline material bonds the system, e. g. calcite, dislocations may be present. In any case, based on energy arguments, the scale of the dislocation mechanism is the correct scale for the mechanism of nonlinear response (e. g. [10]).

CONCLUSIONS

The above models of mesoscopic nonlinearity can be considered as basic for the description of nonlinear acoustic properties of rocks and other consolidated granular materials. They permit an explanation of strong nonlinearity and also some non-classical dependencies observed in experiments. At the same time, reality is still much more complicated. The physical theory of mesoscopic nonlinearity in real materials is a work in progress.

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