Hysteresis and the Dynamic Elasticity of Consolidated Granular Materials

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Quasistatic elasticity measurements on rocks show them to be strikingly nonlinear and to have elastic hysteresis with end point memory. When the model for this quasistatic elasticity is extended to the description of nonlinear dynamic elasticity the elastic elements responsible for the hysteresis dominate the behavior. Consequently, in a resonant bar, driven to nonlinearity, the frequency shift and the attenuation are predicted to be nonanalytic functions of the strain field. A resonant bar experiment yielding results in substantial qualitative and quantitative accord with these predictions is reported. [S0031-9007(99)08968-1]

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Rocks have extreme dynamic elasticity; their velocity of sound changes by a factor of 2 under modest pressure change [1,2] and unusual quasistatic elasticity; their quasistatic stress-strain equation of state has hysteresis with end point memory [3]. These properties make rocks interesting as members in the sequence of elastic systems with increasing cohesiveness (sand, soil, rock, grain) and interesting as members of the class of hysteretic systems that possess memory [4].

Quasistatic stress-strain measurements on rocks involve large strains ($\epsilon \approx 10^{-3}$) and low frequencies ($f \approx 10^{-2}$ Hz) [5]. Dynamic nonlinear elasticity measurements involve very small strains ($\epsilon \approx 10^{-8}$) and high frequencies ($f \approx 10^{4}$ Hz) [1]. A mean field theory, using a Preisach description of hysteretic elastic elements [6,7], gives a sensible picture of the observed quasistatic behavior [8,9]. When this theory is extended to describe dynamic elasticity a number of unusual predictions result [10,11]. The purpose of this paper is to report measurements of dynamic elasticity on a Berea sandstone. These measurements confirm the predictions of the theory and establish its usefulness in describing the quasistatic and dynamic elasticity of consolidated materials over a large strain/time scale range, $10^{-8} < \epsilon < 10^{-3}$ and $10^{-2} < f < 10^{4}$ Hz.

Quasistatic stress-strain measurements on rock, e.g., a Berea sandstone, show striking nonlinearity and hysteresis with end point memory [5]. These observations have a theoretical explanation in terms of a new model for finding the equation of state of consolidated materials [8,9,12]. The new model has been extended to describe nonlinear elastic wave propagation [10] and the case at hand, resonant bar measurements [11,13]. A series of predictions results.

(i) The resonant frequency should have a frequency shift that is linear in the *magnitude* of the strain field, U, i.e.,

$$\frac{\Delta f}{f_0} = C_f |U| . \tag{1}$$

(ii) $1/Q$, a measure of the attenuation, should depart from $1/Q_0$ linearly in the magnitude of the strain field, i.e.,

$$1/Q - 1/Q_0 = C_Q |U| , \tag{2}$$

with $C_Q/C_f = 3\pi/8 \approx 1$ and $C_f = \alpha$. (The constant α, which characterizes the hysteretic elastic elements in the rock, is found from the analysis of quasistatic data, $\alpha \approx 10^3$ for a Berea sandstone [14].)

The sample was a long, thin, cylindrical rod of Berea sandstone, 6 cm $\times$ 30 cm, that had a low amplitude resonance at $f_0 \approx 2880$ Hz with a Q, denoted Q_0, of approximately 170. It was mounted in a sample chamber, evacuated to $\approx$20 mTorr, and temperature controlled to 0.1 °C at approximately room temperature. (The data reported below were taken after the sample had been under vacuum and temperature control for about 3 months.) A PZT-4 piezoelectric disk, having brass inertial backload, was epoxied to one end of the cylinder and acted as the source. It was driven in discrete frequency steps from an HP3325B function generator through a Crown Reference I audio amplifier. The acceleration of the opposite end of the bar was detected with a B&K 8309 accelerometer/B&K 2635 charge amplifier. A reference signal from the function generator and the detected signal were fed to a E&G 5302 lock-in amplifier. The system was capable of measuring accelerations down to 10^{-2} cm/sec^2 or strains of order 10^{-10}. A digital oscilloscope was used to monitor time series and/or spectral content. Sweep rates, frequency steps, voltage steps, and data storage are all controlled by an Apple PowerPC running LABVIEW. (Linearity of the electronics, transducer bond, and the complete measurement system were carefully checked by methods described in previous literature [2].)

An up/down frequency sweep is conducted at fixed source voltage, e.g., V_1. It consists of a sweep up through the frequency sequence $f_1, \ldots, f_N$ followed by a sweep

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down through the same frequencies in reverse order. The source voltage is then stepped to V_2 and an up/down frequency sweep is conducted. In a run up/down frequency sweeps are carried out at M approximately evenly spaced voltages from V_1 to V_M followed by up/down frequency sweeps as the voltage sequence is traversed in reverse order. A run is judged to be of good quality when there is good agreement between the up and down frequency sweeps at both visits to all voltages. At each point (f_i, V_j) in a run, both the in phase and out of phase components of the detected acceleration are recorded. The in phase and out of phase acceleration components are converted to displacements and denoted A_{ij} and B_{ij}, respectively. From them we construct $Uij = \sqrt{A_{ij}^2 + B_{ij}^2}$. We take the three $N \times M$ matrices U, A, and B as the data set for a particular run. The average strain field in the bar is proportional to the displacement. Thus we work with the displacement matrices and regard them as equivalent to the corresponding strain matrices and refer to U, A, and B as the strain.

A frequency sweep, at fixed voltage, through the resonance of the bar leads to changes in the amplitude of the detected signal that are typical of a simple oscillator. Accompanying these changes in amplitude are changes in the strain field in the bar. If the elastic properties of the bar depend upon the strain field within it, the resonating system is changing as the sweep proceeds. Peak bending, typical of nonlinear oscillators [2,15], is a manifestation of such internal changes in a resonating system.

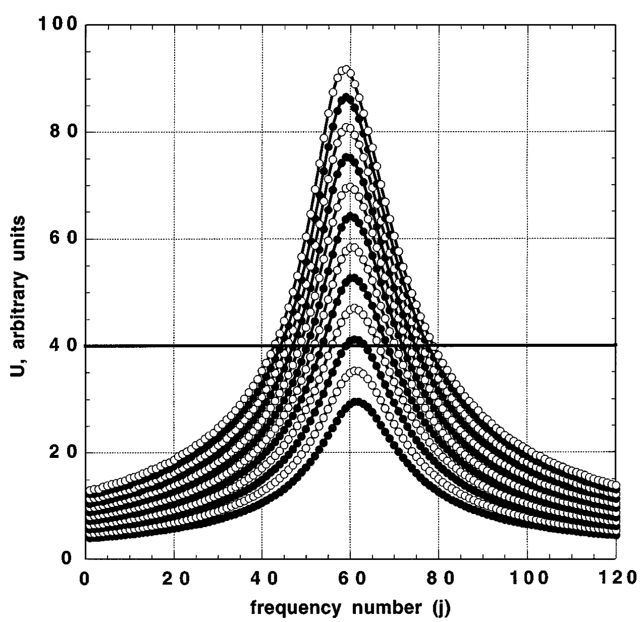

FIG. 1. Strain amplitude as a function of frequency and voltage. The strain is measured at 121 frequencies from 2850 Hz in steps of 0.5 Hz; the step number is indicated on the x axis. The strain amplitude is in arbitrary units with the maximum strain shown corresponding to approximately 3×10^{-7}. Each curve is the data taken at one of twelve constant source voltages. The horizontal line at 40 corresponds to a constant strain cut of the data.

We want to use the response of the rock, brought about by the internal strain field, to study the nonlinear elastic properties of the rock. This is done most cleanly by analysing the experimental data at constant strain. The first step in our data analysis is to contour the strain matrix at constant U, i.e., find trajectories of constant U through U_{ij}. Along each constant U trajectory through (f, V) we also find A and B. To extract information about the elastic properties of the rock along constant U contours we proceed as follows.

(i) At constant strain the rock responds as a simple damped harmonic oscillator [11] for which

$$U = \lambda \frac{V}{(X + iY)} = A + iB, \tag{3}$$

where the oscillator, having natural frequency f_0 and damping Q, is driven at frequency f by a source of voltage strength V. Here, $X = f^2 - f_0^2$, $Y = ff_0/Q$, and λ is the calibration constant that relates the strain to the source voltage. The natural frequency f_0 and Q are constant on a constant U contour but may vary from contour to contour.

(ii) We use Eq. (3) to form ratios of U, A, and B that isolate the parameters f_0 and $1/Q$ that characterize the response

$$L_U \equiv (\lambda^2 V^2/|U|^2) = X^2 + Y^2, \tag{4}$$

$$L_A \equiv (\lambda V^2/|U|^2)\,\frac{A}{V} = X, \tag{5}$$

$$L_B \equiv (\lambda V^2/|U|^2)\,\frac{B}{V} = Y. \tag{6}$$

The quantities on the left-hand sides of these equations are constructed along constant strain trajectories from the experimental U, A, and B matrices. An example is shown in Fig. 2. We emphasize that the data presentation in this figure results simply from manipulating the data in the fashion suggested by the form on the left-hand sides of Eqs. (2)–(4).

(iii) The right-hand sides of Eqs. (2)–(4) further suggest that L_U, L_A, and L_B should be simple polynomials in f. Thus along each constant U contour we fit L_U to $P_U = u_0 + u_2 f^2 + u_4 f^4$, L_A to $P_A = a_0 + a_2 f^2$, and L_B to $P_B = b_0 + b_1 f$. The coefficients $[u_0, u_2, u_4, a_0, a_2, b_0, b_1]$, contant on a constant U contour, vary from contour to contour. It is this variation that tells us about the strain response of the rock, the nonlinearity of the rock.

(iv) From the frequency at which P_A passes through zero and the frequency at which P_U is a minimum, we get two measures of the resonant frequency, f_0, at each strain. Similarly from b_0 and the value of P_U at its minimum we get two measures of the damping, $1/Q$, at each strain.

The results of analysis of the experimental data according to this scheme are shown in Fig. 3. In Fig. 3a

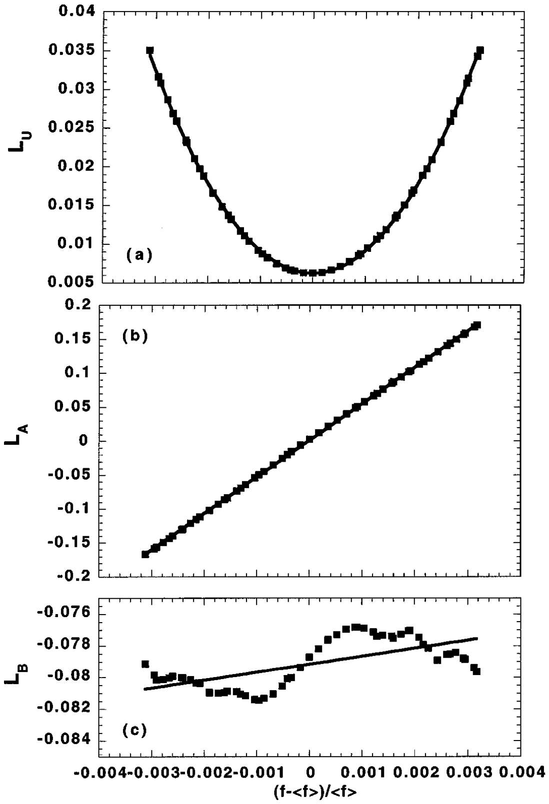

FIG. 2. Behavior along contours of constant strain. The quantities L_U (a), L_A (b), and L_B (c) along contours of constant strain, U, show the frequency dependence of the amplitude, of the in phase component and out of phase component of the strain, Eqs. (4)–(6). The strain dependence of the resonant frequency and of $1/Q$ is found from fitting simple polynomials in f to these curves. The three curves shown here are for the contour at $U = 40$ in Fig. 1.

we show the resonant frequency as a function of strain for three runs that cover overlapping low strain regimes ($10^{-8} \leq \epsilon \leq 3 \times 10^{-7}$, $2850 \leq f \leq 2910$ Hz). The results from each run are shifted from one another on the curve. There is a pair of symbols associated with the data for each run, one of these shows the resonant frequency from the zero of P_A and the other shows the resonant frequency from the minimum of P_U. The near overlap of the two determinations of f_0 is a further check on consistency of the data analysis. [The shift of the three curves (taken several weeks apart) is a consequence of slowly varying shifts in the state of the rock due to temperature and saturation. The very elaborate run acceptance criterion described above was intended as a check that during a run (typically an hour) such shifts were not significant.] The frequency shift corresponds to a softening nonlinearity with

$$f_0(0) - f_0(U) \approx C_f |U|, \tag{7}$$

$C_f = 8 \pm 2 \times 10^{-6}$. In Fig. 3b we show $1/Q$ as a function of strain for the same three runs as in Fig. 3a. Again there is a pair of symbols for each run (an open and

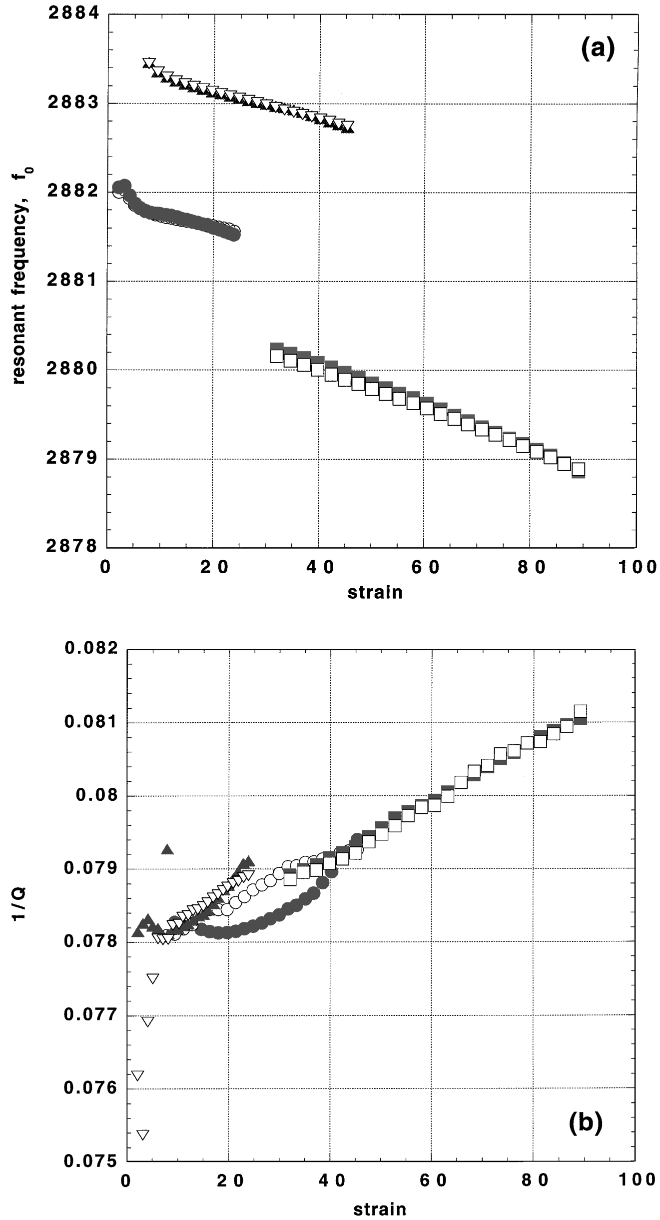

FIG. 3. (a) The resonant frequency as a function of the strain. From the study of the behavior of L_A and L_U at fixed strain there are two determinations of the resonant frequency at that strain. For each of three runs that cover overlapping low strain regimes, $0.095 \leq V \leq 1.97$, $0.47 \leq V \leq 3.74$, $2.34 \leq V \leq 7.47$ ($10^{-8} \leq \epsilon \leq 5 \times 10^{-7}$), $2850 \leq f \leq 2910$ Hz, there is a pair of symbols corresponding to these determinations. The spread from run to run is a slow drift that we are unable to eliminate. (See the discussion of the procedure for accepting a run in the text.) The value of C_f is from the slope of the run with the largest source voltage (shown with open and closed squares). (b) $1/Q$ as a function of strain. From the study of the behavior of L_B and L_U there are two determinations of $1/Q$. For each of three runs there is a pair of symbols corresponding to these determinations. The value of C_Q is from the data shown with open and closed squares.

closed symbol of the same shape) that show consistency. The behavior of $1/Q$ is far noisier than the behavior of the frequency shift but $1/Q$ seems not to be subject to the uncontrolled slow drift seen in f_0. From these data we find

$$\frac{1}{Q(U)} - \frac{1}{Q_0} \approx C_Q|U|, \tag{8}$$

$C_Q = 2 \pm 1 \times 10^{-6}$. The frequency shift and $1/Q$ are proportional to the magnitude of the strain field with $C_Q/C_f \approx 0.25$. It remains to compare the magnitude of C_f and C_Q to the predictions from the analysis of quasistatic data. The strain scale we have used until now, e.g., in Figs. 1–3, has been in arbitrary units. Using the measured accelerations and the length of the resonant bar, we find that the maximum strain explored is $\epsilon \approx 3 \times 10^{-7}$. Thus the observed frequency shift is $(\Delta f/f_0)_X \approx 2.7 \times 10^3 \epsilon$.

In the theory of the response of the resonant bar [11], the frequency shift is given by $\Delta f/f_0 \approx \gamma\epsilon$, $\gamma = K_0\alpha/A_0$, where K_0 is the compressibility, and α and A_0 are coefficients determined from the quasistatic data; A_0 and α measure the nonhysteretic and hysteretic components of the elastic response, respectively. Recently Guyer *et al.* [14] have analyzed an extensive quasistatic data set on a Berea sandstone with the result $K_0 \approx 10$ GPa, $A_0 \approx 10^{-4}$ GPa^{-1}, $\alpha \approx 10^{-2}$ GPa^{-2}. This leads to

$$\left(\frac{\Delta f}{f_0}\right)_T \approx 1000\epsilon\,. \tag{9}$$

The order of magnitude agreement between $(\Delta f/f_0)_X$ and $(\Delta f/f_0)_T$ is gratifying.

We take the package of results, the linear dependence of $\Delta f/f$ and $1/Q$ on $|U|$, the quantitative relationship of C_f to C_Q, and the quantitative relationship of C_f to quasistatic parameters as strong evidence that the picture of dynamic nonlinear elasticity we are testing has substantial validity. The alternative, the traditional theory of nonlinear elasticity [16], leads to a frequency shift that is quadratic in the strain field with $\Delta f/f_0 \approx 10^{-10}$ at $\epsilon \approx 3 \times 10^{-7}$, i.e., to results having neither a qualitative nor a quantitative relationship to the experimental findings. The picture of nonlinear elasticity dominated by an assemblage of hysteretic elastic elements provides a synthesis of quasistatic and dynamic elastic behavior over a substantial strain/frequency range. Quite possibly this picture can be usefully employed in understanding some of the properties of sand, soil, concrete, ceramics, etc., systems that in are in some circumstances similar to rocks.

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