Slow elastic dynamics in a resonant bar of rock

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Abstract. Recent resonant bar experiments on Berea sandstone show that nonlinear excitation of the sample excites a slow dynamics with a time scale many orders of magnitude longer than the excitation period, $2\pi/\omega$. That is, a nonlinear resonant frequency decays to the linear resonant frequency long after the high amplitude drive has been turned off. We postulate a phenomenological theory of slow nonlinear dynamics in the context of a resonant bar experiment. The normalized elastic modulus of the resonant bar is allowed to be nonlinear and time dependent. The nonlinear terms are derived from a model of elasticity in rocks that includes anharmonic and hysteretic contributions. We use this theory to explain the experimental results. We find an explanation for the slow relaxation of the experimental resonant frequency using an anharmonic contribution to the modulus that responds instantaneously to a disturbance, and a contribution derived from elastic hysteresis that displays slow dynamics. We suggest an acoustic NMR-type experiment to explore slow nonlinear dynamics.

Introduction

Wave propagation in rocks has often been described by the time tested linear theory of elasticity [Landau and Lifshitz, 1959]. At least three empirical findings suggest the need for a broader strategy in the experimental and theoretical investigation of wave propagation. (1) The cubic and quartic anharmonicities in rocks are enormous relative to those of materials successfully described by traditional theory [Johnson and Rasolofosaon, 1996]. (2) Quasi-static stress-strain measurements on rocks display hysteresis with discrete memory [Boitnott, 1993; Holcomb, 1981]. (3) A slow dynamics has been observed in the behavior of the elastic modulus of Berea sandstone [TenCate and Shankland, 1997].

The purpose of this letter is to introduce a phenomenology for elastic wave propagation in rocks. This phenomenology is developed in the context of resonant bar experiments, for these experiments are characterized by a high degree of precision and control. This phenomenology respects the following empirical facts.

1. Experiment shows that the velocities of sound in rock can vary by a factor of approximately two over the pressure range (0,100) MPa [Bourbie et al., 1987]. Thus the coefficient that measures cubic anharmonicity is of order $10^3$. This coefficient is of order 10 for normal (single crystal) materials, e.g., SiO$_2$ [Ashcroft and Mermin, 1976]. As a consequence the stress field that determines the propagation of elastic waves is of the form

$$\sigma_A(x) = K_0 \left[ 1 + \beta \frac{\partial u}{\partial x} + \delta \left( \frac{\partial u}{\partial x} \right)^2 + \cdots \right] \frac{\partial u}{\partial x},$$

where $\partial u/\partial x$ is the strain field, $K_0$ is the linear modulus, and $\beta$ and $\delta$ are measures of the cubic and quartic anharmonicities [Landau and Lifshitz, 1959; Van Den Abeele et al., 1997].

2. The hysteresis with discrete memory, seen in quasi-static stress-strain measurements, is described by hysteretic elastic elements in the rock [McCall and Guyer, 1994]. The behavior of these elastic elements is dependent on the elastic history of each point in the rock. In the context of a resonant bar experiment the hysteretic elastic elements make a contribution to the stress of the form

$$\sigma_H = K_0 \alpha \left( \Delta \varepsilon \pm \frac{\partial u}{\partial x} \right) \frac{\partial u}{\partial x},$$

where $\Delta \varepsilon$ is maximum strain excursion, $\alpha$ is the strength of the hysteresis, and the plus sign corresponds to the modulus during increasing strain, the minus sign to decreasing strain [Van Den Abeele et al., 1997].

Theory

Consider a resonant bar experiment in which a bar of length $L$ (a cylindrical rock sample long compared to its width) is driven at one end with a force at frequency $\omega$. The acceleration response at the other end of the bar, at the driving frequency $\omega$, is detected. Usually the driving frequency $\omega$ is swept through the linear resonant frequency of the bar $\omega_0 = c/L$, where $c$ is the velocity of sound. Take the stress field in the bar to be to be the sum of (1) and (2), $\sigma = \sigma_A + \sigma_H$. Then the equation of motion for the displacement field $u$ is

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{\tau_0} \frac{\partial u}{\partial t} - \frac{1}{\tau} \frac{\partial u}{\partial x} + \frac{\varepsilon^2}{\tau} \left( \frac{\partial u}{\partial x} \right)^2 + \alpha \left( \Delta \varepsilon \pm \frac{\partial u}{\partial x} \right) \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + \cdots \right] \frac{\partial^2 u}{\partial x^2} = f \cos \omega t,$$

where $\tau_0$ is the characteristic time of the hysteresis, $\tau$ is the characteristic time of the anharmonic contribution, $\varepsilon$ is the ratio of anharmonic energy to kinetic energy, and $\alpha$ is the intensity of the hysteresis. The equation of motion is

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{\tau_0} \frac{\partial u}{\partial t} - \frac{1}{\tau} \frac{\partial u}{\partial x} + \frac{\varepsilon^2}{\tau} \left( \frac{\partial u}{\partial x} \right)^2 + \alpha \left( \Delta \varepsilon \pm \frac{\partial u}{\partial x} \right) \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + \cdots \right] \frac{\partial^2 u}{\partial x^2} = f \cos \omega t, \tag{3}$$
where $r_0$ characterizes a phenomenological damping, and $f$ is the force per unit mass. In resonance, the dominant terms in the equation of motion are those that feed back into the driving frequency. Thus to first order, the terms proportional to $\beta$ and $\pm \alpha \partial u/\partial x$ are negligible.

Using the Ritz averaging method on the resulting lumped element equation [Timoshenko et al., 1974] leads to an implicit equation for the amplitude of the displacement field in the bar,

$$A = \frac{F}{\sqrt{(\Omega^2 - 1 + \lambda_1 A + \lambda_2 A^2)^2 + \frac{\Omega^2}{Q^2}}},$$  

(4)

where $\Omega = \omega/\omega_0$, $Q = r_0 \omega_0$, and $\lambda_1$ and $\lambda_2$ are coefficients proportional to $\alpha$ and $\delta$ respectively ($\lambda_1$ and $\lambda_2$ are positive when $\alpha$ and $\delta$ are negative, as is the case for strain softening in rocks). In this equation the displacement field and the force are dimensionless, $F = f/L^2 \omega_0$, and $A = a/L$, where $a$ is the displacement amplitude. Thus, from this point forward all quantities are dimensionless. The terms $K = 1 - \lambda_1 A - \lambda_2 A^2$ are referred to as the normalized (nonlinear) elastic modulus of the bar. This elastic modulus involves a first order contribution proportional to the displacement amplitude $A$ and the strength of the hysteresis $\alpha$, and a second order contribution proportional to $A^2$ and the strength of the quartic anharmonicity $\delta$. According to (4), for $Q \gg 1$, the maximum displacement amplitude occurs when the resonant frequency shift $1 - \Omega^2$ is given by

$$1 - \Omega^2 \approx \lambda_1 A + \lambda_2 A^2.$$  

(5)

The experimental observation of a frequency shift that is first order in $A$ demonstrates that the hysteretic elastic elements in the rock contribute to the response of the rock in a resonance experiment [Guyer et al., 1995].

Equation (4) describes the long time behavior of the amplitude, that is, the nonlinear contributions to the resonant frequency have no time dependence. However, Tencate and Shankland [1997] have demonstrated that there are long time scales involved in the elastic response of a rock. To describe the slow dynamics of Tencate and Shankland [1997] we postulate the coupled system of equations

$$A = \frac{F}{\sqrt{(\Omega^2 - 1 + \Delta K_1 + \Delta K_2)^2 + \frac{\Omega^2}{Q^2}}},$$  

(6)

$$\frac{d\Delta K_1}{dt} = -\frac{\Delta K_1}{\tau_1} + \frac{\lambda_1}{\tau_1} A,$$  

(7)

$$\frac{d\Delta K_2}{dt} = -\frac{\Delta K_2}{\tau_2} + \frac{\lambda_2}{\tau_2} A^2.$$  

(8)

These equations are assumed to describe the behavior of the amplitude on time scale long compared to the time for the amplitude to equilibrate with the driving force and the attenuation mechanism, i.e., $Q/\omega$. The relaxation times $\tau_1$ and $\tau_2$ are the times over which the system has memory of its past elastic state. For example, a solution to (7) is

$$\Delta K_1(t) = \Delta K_1(0) e^{-t/\tau_1} + \frac{\lambda_1}{\tau_1} \int_0^t A(t') e^{(t'-t)/\tau_1} dt'.$$  

(9)

When the bar has been in a particular elastic state for a time long compared to $\tau_1$, $A$ is independent of $t$, and $\Delta K_1 \sim \lambda_1 A$. Thus if the bar is driven at fixed $F$ and $\Omega$ for a time long compared to $\tau_1$ and $\tau_2$, $A$ is given by (4). If at fixed $F$ the driving frequency is swept rapidly compared to both $\tau_1$ and $\tau_2$, then the elastic modulus is dependent on the average of $A$ over times of order $\tau_1$ and $\tau_2$, as in (9).

Consider the behavior of $A$ when the driving time is much greater than $\tau_1$ and $\tau_2$ for each $\Omega$. Then $\Delta K_1 = \lambda_1 A$, $\Delta K_2 = \lambda_2 A^2$, and (6) reduces to (4). In Figure 1 we show $A/F$ from (4) as a function of $\Omega$ for $\lambda_1 = 0.5$, $\lambda_2 = 1.0$, and $Q = 5$. The figure is a series of resonance curves for 25 values of $F$ distributed logarithmically between 0.001 and 0.07. As $F$ increases, the shift of the resonant frequency from the linear resonant frequency, $\Omega_0 = 0.99$, becomes very pronounced. In Figure 2 the shift of the resonant frequency from $\Omega_0$ is plotted as a function of the amplitude at resonance. Note that the frequency shift is linear in the amplitude at low peak amplitudes in accordance with (5) in the limit of small frequency shifts; i.e., $\delta \Omega = \Omega_0 - \Omega \ll 1$ leads to $2 \delta \Omega \approx \lambda_1 A + \lambda_2 A^2$. Thus we see that the contribution to the nonlinear modulus due to hysteresis dominates the resonant frequency shift at low amplitude.

Next we consider the dynamic response of the system. When the driving force $F$ in (6) is so small that $\lambda_1 A + \lambda_2 A^2 \ll 1$, the right hand side (RHS) of (6) is es-

Figure 1. Amplitude scaled by the driving force $A/F$ is plotted as a function of frequency $\Omega$ for 25 values of $F$.

Figure 2. Resonant frequency shift $\delta \Omega = \Omega_0 - \Omega$, as a function of dimensionless amplitude at resonance. The straight line has slope 1.
sentially independent of both \( A \) and time, and (7) and (8) are unnecessary. However for \( F \) sufficiently large, the nonlinear terms on the RHS of (6) are important; the amplitudes that determine \( \Delta K_1 \) and \( \Delta K_2 \) are averages of the amplitudes at earlier moments in time. Thus in a resonant bar experiment in which the frequency is changed, the response at any moment of time depends on the rate of frequency change. Since the largest amplitude changes occur with changing frequency near the resonance, for an experiment to be carried out slowly it must pass the resonance region slowly compared to \( \tau_1 \) (assuming \( \tau_1 > \tau_2 \)), i.e.,

\[
\tau_1 \left( \frac{d\omega}{dt} \right)_x \ll \Delta \omega \approx \frac{\omega_0}{Q}
\]

where \( (d\omega/dt)_x \) is the rate at which the frequency is swept and \( \Delta \omega \) is the width of the resonance curve at half power. We define the time \( \tau_x \) to characterize an experimental frequency protocol

\[
\tau_x = \frac{1}{Q} \left( \frac{dt}{d\omega} \right)_x.
\]

If \( \tau_x > \tau_1 \), then \( A \) is given by (4). If \( \tau_x \) is of order \( \tau_1 \), then the slow dynamics becomes apparent. This is illustrated in Figure 3 where the amplitude from solution to (6)-(8) is plotted as a function of time step as the driving frequency \( \Omega \) evolves. We have chosen \( Q = 5, \lambda_1 = 0.5, \lambda_2 = 0, \tau_1 = 10, A(t = 0) = 0 \) and \( F = 0.050 \). The resonant frequency is \( \Omega_0 \approx 0.87 \). The conditions of this numerical experiment (\( \tau_x \) of order \( \tau_1 \)) are the same as those for the experimental result shown in Figure 6 of TenCate and Shankland [1997]; our numerical experiment is in agreement with parts (b) and (d). Note that when the slow dynamics is operating the resonance shifts in height and width according to whether \( \Omega \) is swept up or down. Similar features are observed in the experimental results.

The sequence of linear probes is shown in Figure 4. Note, when the frequency sweep is stopped after having passed through the resonance from above (below), the amplitude response decays to a lower (higher) steady state value. These qualitative features are in agreement with experimental observations.
The instantaneous decay in resonant frequency upon changing $F$ from $F_{\text{max}}$ to 0.001 is the difference between $Q_0 = 0.99$ and (a) the linear resonance frequency shifts as a function of the amplitude for several values $K = 1 - \lambda_1 A - \lambda_2 A^2 = 0.648$, and the resonance frequency is $\Omega \approx 0.8 \pm 0.05^2 \approx K$. Immediately after the drive is turned down, $t = 202$, the amplitude relaxes to $A \approx 0.005$, and the resonant frequency shifts to $\Omega \approx 0.89$. The anharmonic contribution to the elastic modulus has decayed away ($\tau_2 = 0$), while the hysteretic contribution remains. Thus $K$ snaps from 0.648 to $K = 1 - \lambda_1 \times 0.394 - \lambda_2 \times (0.005)^2 = 0.803$. The hysteretic component of the elastic modulus relaxes slowly to the low amplitude value, $A \approx 0.005$, and the resonant frequency evolves toward $\Omega_0 = 0.99$.

In Figure 5 we have plotted the resonant frequency shifts as a function of the amplitude for several values of $F_{\text{max}}$. Otherwise, the protocol is identical to that described for Figure 4. The frequency shifts are the difference between $\Omega_0 = 0.99$ and (a) the linear resonant frequency at $t = 198$ (circles), and (b) the linear resonant frequency at $t = 202$ (filled circles). The dependence of the latter frequency shift is linear in the pumping amplitude response and proportional to $\lambda_1$. The instantaneous decay in resonant frequency upon changing $F$ from $F_{\text{max}}$ to 0.001 is the difference between the open circles and the filled circles. The difference between these two frequencies increases as $F_{\text{max}}$ increases since the second order (anharmonic) contribution to the elastic modulus increases. These results are in accord with the preliminary findings of TenCate [1997].

**Conclusion**

In this paper we have introduced a phenomenological model of the dynamic nonlinear elastic response of a rock. The model has two distinct kinds of nonlinearity, a traditional anharmonic contribution, and a contribution derived from elastic hysteresis. Each kind of nonlinearity is allowed to have a slow dynamics. A simple explanation of the experimental findings of TenCate and Shankland [1997], in which a slow dynamics makes itself known, is provided by the model.

The existence of a slow dynamics in strain response means that in any experiment that disturbs the rock beyond linearity there will be features of the time history of the disturbance in the response to the disturbance. The enormous separation between time scales (the period of the fundamental is less than 0.01 sec; the time scale for slow dynamics is greater than 100 sec) suggests that many features of the slow dynamics in the nonlinearity may be probed using NMR-style experimental protocols. A preliminary illustration of such a protocol yields results similar to those found in experiment.

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