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Slow elastic dynamics in a resonant bar of rock

R. A. Guyer
Department of Physics and Astronomy, University of Massachusetts, Amherst

K. R. McCall
Department of Physics, University of Nevada, Reno

K. Van Den Abeele
Division of Earth and Environmental Sciences, Los Alamos National Laboratory, New Mexico

Abstract. Recent resonant bar experiments on Berea sandstone show that nonlinear excitation of the sample excites a slow dynamics with a time scale many orders of magnitude longer than the excitation period, $2\pi/\omega$. That is, a nonlinear resonant frequency decays to the linear resonant frequency long after the high amplitude drive has been turned off. We postulate a phenomenological theory of slow nonlinear dynamics in the context of a resonant bar experiment. The normalized elastic modulus of the resonant bar is allowed to be nonlinear and time dependent. The nonlinear terms are derived from a model of elasticity in rocks that includes anharmonic and hysteretic contributions. We use this theory to explain the experimental results. We find an explanation for the slow relaxation of the experimental resonant frequency using an anharmonic contribution to the modulus that responds instantaneously to a disturbance, and a contribution derived from elastic hysteresis that displays slow dynamics. We suggest an acoustic NMR-type experiment to explore slow nonlinear dynamics.

Introduction

Wave propagation in rocks has often been described by the time tested linear theory of elasticity [*Landau and Lifshitz*, 1959]. At least three empirical findings suggest the need for a broader strategy in the experimental and theoretical investigation of wave propagation. (1) The cubic and quartic anharmonicities in rocks are enormous relative to those of materials successfully described by traditional theory [*Johnson and Rasolofosaon*, 1996]. (2) Quasi-static stress-strain measurements on rocks display hysteresis with discrete memory [*Boitnott*, 1993; *Holcomb*, 1981]. (3) A slow dynamics has been observed in the behavior of the elastic modulus of Berea sandstone [*TenCate and Shankland*, 1997].

The purpose of this letter is to introduce a phenomenology for elastic wave propagation in rocks. This phenomenology is developed in the context of resonant bar experiments, for these experiments are characterized by a high degree of precision and control. This phenomenology respects the following empirical facts.

1. Experiment shows that the velocities of sound in rock can vary by a factor of approximately two over the pressure range (0,100) MPa [*Bourbie et al.*, 1987]. Thus the coefficient that measures cubic anharmonicity is of order 10^3. This coefficient is of order 10 for normal (single crystal) materials, e.g., SiO_2 [*Ashcroft and Mermin*, 1976]. As a consequence the stress field that determines the propagation of elastic waves is of the form

$$\sigma_A(x) = K_0 \left[1 + \beta \frac{\partial u}{\partial x} + \delta \left(\frac{\partial u}{\partial x} \right)^2 + \cdots \right] \frac{\partial u}{\partial x}, \quad (1)$$

where $\partial u/\partial x$ is the strain field , K_0 is the linear modulus, and β and δ are measures of the cubic and quartic anharmonicities [*Landau and Lifshitz*, 1959; *Van Den Abeele et al.*, 1997].

2. The hysteresis with discrete memory, seen in quasistatic stress-strain measurements, is described by hysteretic elastic elements in the rock [*McCall and Guyer*, 1994]. The behavior of these elastic elements is dependent on the elastic history of each point in the rock. In the context of a resonant bar experiment the hysteretic elastic elements make a contribution to the stress of the form

$$\sigma_H = K_0 \alpha \left(\Delta\epsilon \pm \frac{\partial u}{\partial x} \right) \frac{\partial u}{\partial x}, \quad (2)$$

where $\Delta\epsilon$ is maximum strain excursion, α is the strength of the hysteresis, and the plus sign corresponds to the modulus during increasing strain, the minus sign to decreasing strain [*Van Den Abeele et al.*, 1997].

Theory

Consider a resonant bar experiment in which a bar of length L (a cylindrical rock sample long compared to its width) is driven at one end with a force at frequency ω. The acceleration response at the other end of the bar, at the driving frequency ω, is detected. Usually the driving frequency ω is swept through the linear resonant frequency of the bar $\omega_0 = \pi c/L$, where c is the velocity of sound. Take the stress field in the bar to be to be the sum of (1) and (2), $\sigma = \sigma_A + \sigma_H$. Then the equation of motion for the displacement field u is

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{\tau_0}\frac{\partial u}{\partial t} - c^2 \left[1 + \beta \frac{\partial u}{\partial x} + \delta \left(\frac{\partial u}{\partial x} \right)^2 + \alpha \left(\Delta\epsilon \pm \frac{\partial u}{\partial x} \right) + \cdots \right] \frac{\partial^2 u}{\partial x^2} = f \cos \omega t, \quad (3)$$

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where τ_0 characterizes a phenomenological damping, and f is the force per unit mass. In resonance, the dominant terms in the equation of motion are those that feed back into the driving frequency. Thus to first order, the terms proportional to β and $\pm\alpha\partial u/\partial x$ are negligible.

Using the Ritz averaging method on the resulting lumped element equation [*Timoshenko et al.*, 1974] leads to an implicit equation for the amplitude of the displacement field in the bar,

$$A = \frac{F}{\sqrt{(\Omega^2 - 1 + \lambda_1 A + \lambda_2 A^2)^2 + \frac{\Omega^2}{Q^2}}}, \quad (4)$$

where $\Omega = \omega/\omega_0$, $Q = \tau_0\omega_0$, and λ_1 and λ_2 are coefficients proportional to α and δ respectively (λ_1 and λ_2 are positive when α and δ are negative, as is the case for strain softening in rocks). In this equation the displacement field and the force are dimensionless, $F = f/L\omega_0^2$, and $A = a/L$, where a is the displacement amplitude. Thus, from this point forward all quantities are dimensionless. The terms $K = 1 - \lambda_1 A - \lambda_2 A^2$ are referred to as the normalized (nonlinear) elastic modulus of the bar. This elastic modulus involves a first order contribution proportional to the displacement amplitude A and the strength of the hysteresis α, and a second order contribution proportional to A^2 and the strength of the quartic anharmonicity δ. According to (4), for $Q \gg 1$, the maximum displacement amplitude occurs when the resonant frequency shift $1 - \Omega^2$ is given by

$$1 - \Omega^2 \approx \lambda_1 A + \lambda_2 A^2. \quad (5)$$

The experimental observation of a frequency shift that is first order in A demonstrates that the hysteretic elastic elements in the rock contribute to the response of the rock in a resonance experiment [*Guyer et al.*, 1995].

Equation (4) describes the long time behavior of the amplitude, that is, the nonlinear contributions to the resonant frequency have no time dependence. However, *Tencate and Shankland* [1997] have demonstrated that there are long time scales involved in the elastic response of a rock. To describe the slow dynamics of *Tencate and Shankland* [1997] we postulate the coupled system of equations

$$A = \frac{F}{\sqrt{(\Omega^2 - 1 + \Delta K_1 + \Delta K_2)^2 + \frac{\Omega^2}{Q^2}}}, \quad (6)$$

$$\frac{d\Delta K_1}{dt} = -\frac{\Delta K_1}{\tau_1} + \frac{\lambda_1}{\tau_1}A, \quad (7)$$

$$\frac{d\Delta K_2}{dt} = -\frac{\Delta K_2}{\tau_2} + \frac{\lambda_2}{\tau_2}A^2. \quad (8)$$

These equations are assumed to describe the behavior of the amplitude on time scale long compared to the time for the amplitude to equilibrate with the driving force and the attenuation mechanism, i.e., Q/ω. The relaxation times τ_1 and τ_2 are the times over which the system has memory of its past elastic state. For example, a solution to (7) is

$$\Delta K_1(t) = \Delta K_1(0)e^{-t/\tau_1} + \frac{\lambda_1}{\tau_1}\int_0^t A(t')e^{(t'-t)/\tau_1}dt'. \quad (9)$$

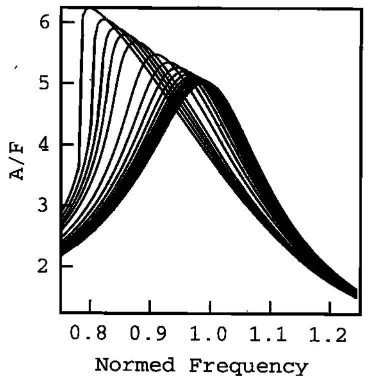

Figure 1. Amplitude scaled by the driving force A/F is plotted as a function of frequency Ω for 25 values of F.

When the bar has been in a particular elastic state for a time long compared to τ_1, A is independent of t, and $\Delta K_1 \sim \lambda_1 A$. Thus if the bar is driven at fixed F and Ω for a time long compared to τ_1 and τ_2, A is given by (4). If at fixed F the driving frequency is swept rapidly compared to both τ_1 and τ_2, then the elastic modulus is dependent on the average of A over times of order τ_1 and τ_2, as in (9).

Consider the behavior of A when the driving time is much greater than τ_1 and τ_2 for each Ω. Then $\Delta K_1 = \lambda_1 A$, $\Delta K_2 = \lambda_2 A^2$, and (6) reduces to (4). In Figure 1 we show A/F from (4) as a function of Ω for $\lambda_1 = 0.5$, $\lambda_2 = 1.0$, and $Q = 5$. The figure is a series of resonance curves for 25 values of F distributed logarithmically between 0.001 and 0.07. As F increases, the shift of the resonant frequency from the linear resonant frequency, $\Omega_0 = 0.99$, becomes very pronounced. In Figure 2 the shift of the resonant frequency from Ω_0 is plotted as a function of the amplitude at resonance. Note that the frequency shift is linear in the amplitude at low peak amplitudes in accordance with (5) in the limit of small frequency shifts; i.e., $\delta\Omega = \Omega_0 - \Omega \ll 1$ leads to $2\delta\Omega \approx \lambda_1 A + \lambda_2 A^2$. Thus we see that the contribution to the nonlinear modulus due to hysteresis dominates the resonant frequency shift at low amplitude.

Next we consider the dynamic response of the system. When the driving force F in (6) is so small that $\lambda_1 A + \lambda_2 A^2 \ll 1$, the right hand side (RHS) of (6) is es-

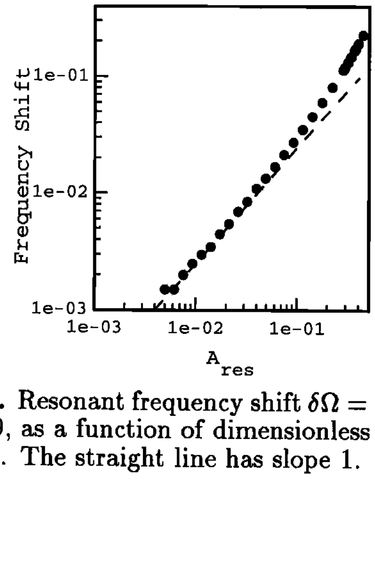

Figure 2. Resonant frequency shift $\delta\Omega = \Omega_0 - \Omega$, where $\Omega_0 = 0.99$, as a function of dimensionless amplitude at resonance. The straight line has slope 1.

sentially independent of both A and time, and (7) and (8) are unnecessary. However for F sufficiently large, the nonlinear terms on the RHS of (6) are important; the amplitudes that determine ΔK_1 and ΔK_2 are averages of the amplitudes at earlier moments in time. Thus in a resonant bar experiment in which the frequency is changed, the response at any moment of time depends on the rate of frequency change. Since the largest amplitude changes occur with changing frequency near the resonance, for an experiment to be carried out slowly it must pass the resonance region slowly compared to τ_1 (assuming $\tau_1 > \tau_2$), i.e.,

$$\tau_1 \left(\frac{d\omega}{dt}\right)_X \ll \Delta\omega \approx \frac{\omega_0}{Q} \tag{10}$$

where $(d\omega/dt)_X$ is the rate at which the frequency is swept and $\Delta\omega$ is the width of the resonance curve at half power. We define the time τ_X to characterize an experimental frequency protocol

$$\tau_X = \frac{1}{Q}\left(\frac{dt}{d\Omega}\right)_X . \tag{11}$$

If $\tau_X \gg \tau_1$, then A is given by (4). If τ_X is of order τ_1, then the slow dynamics becomes apparent. This is illustrated in Figure 3 where the amplitude from solution to (6)–(8) is plotted as a function of time step as the driving frequency Ω evolves. We have chosen $Q = 5$, $\lambda_1 = 0.5$, $\lambda_2 = 0$, $\tau_1 = 10$, $A(t = 0) = 0$ and $F = 0.050$. The resonant frequency is $\Omega_0 \approx 0.87$. The conditions of this numerical experiment (τ_X of order τ_1) are the same as those for the experimental result shown in Figure 6 of *TenCate and Shankland* [1997]; our numerical experiment is in agreement with parts (b) and (d). Note that when the slow dynamics is operating the resonance shifts in height and width according to whether Ω is swept up or down. Similar features are observed in the experimental results.

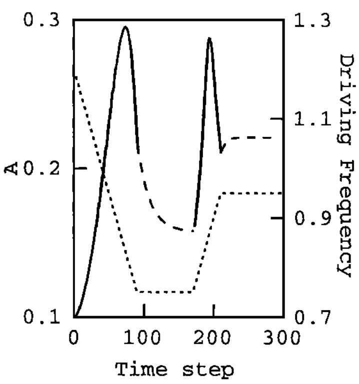

Figure 3. Amplitude as a function of time step for fixed drive and the driving frequency protocol shown. The amplitude is the solid and dashed line; the driving frequency Ω is the dotted line. This Ω protocol is similar to that employed by *TenCate and Shankland* [1997] to generate their Figure 6. Note, when the frequency sweep is stopped after having passed through the resonance from above (below), the amplitude response decays to a lower (higher) steady state value. These qualitative features are in agreement with experimental observations.

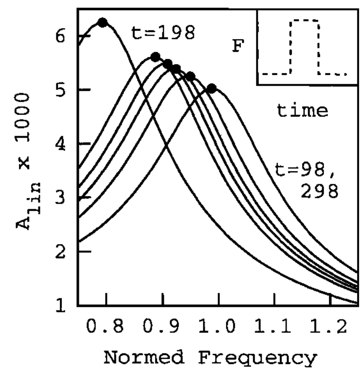

Figure 4. Linear probe amplitude as a function of frequency for 8 linear probes of the system as described in the text. The linear response at $t = 98$ and $t = 298$ is the far right curve. The linear response at $t = 198$ is far left curve. The four intermediate curves are the linear response at $t = 202$, 204, 206, and 210 (left to right). The location of the resonance on each curve is indicated by a filled circle. The inset is a schematic of the driving force protocol described in the text.

The elastic state of the rock is set by the strain field in the rock. The result in Figure 3 is complex because the calculation (like experiment) involves a frequency sweep at fixed force and as a consquence, varying strain field. Just as specification of the stress as a function of time is an integral part of characterizing quasi-static stress-strain measurements [*McCall and Guyer,* 1994], specification of F and Ω as a function of time is an integral part of characterizing dynamic measurements.

A frequency sweep at fixed drive is a good test of the elastic state of the rock provided that (1) the frequency sweep is fast enough that the elastic state of the rock does not change markedly during the sweep and (2) the frequency sweep does not itself significantly modify the elastic state of the rock. Thus we envision an NMR-style elastic experiment in which a rapid low amplitude sweep through resonance serves to probe the state of a rock that has been driven at high amplitude and fixed (or slowly varying) frequency.

In Figure 4 we show results of calculations that use a linear frequency sweep as a probe. The rock is excited at fixed frequency $\Omega = 0.95$, and probed at low amplitude using a frequency sweep; the driving amplitude is varied over time. The driving amplitude $F = 10^{-3}$ for $0 < t < 100$, $F = F_{\max} = 0.125$ for $100 < t < 200$, and $F = 10^{-3}$ for $t > 200$, as shown in the inset of Fig. 4. We use $\lambda_1 = 0.5$, $\lambda_2 = 1$, $Q = 5$, $\tau_2 = 0$, and $\tau_1 = 10$. In making our choice of τ_2, we are assuming that the anharmonic nonlinearity responds instantaneously to the driving force. Thus we have $K = 1 - \Delta K_1 - \lambda_2 A^2$ with A and ΔK_1 given by solution of (6) and (7). This system is probed by a linear frequency sweep at $t = 98$ (F low, system stable), at $t = 198$ (F high, system stable), at a sequence of moments in time beyond $t = 200$ (F recently decreased), and at $t = 298$ (F low, system stable).

The sequence of linear probes is shown in Figure 4. The linear probe amplitude is plotted as a function of Ω for seven resonance curves. From left to right the resonant curves were taken at $t = 198$, 202, 204, 206, 210, and 298. The far right curve is identically the resonance curves at $t = 98$ and $t = 298$; this curve is the linear resonance curve of the rock with resonance frequency

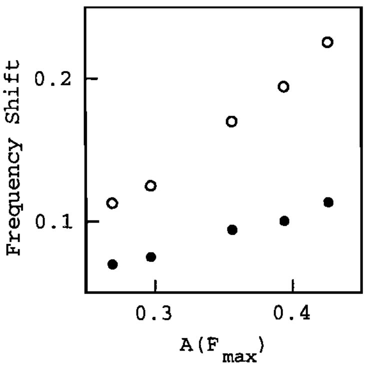

Figure 5. Frequency shift as a function of amplitude during and after a sustained high amplitude drive. From left to right, $F_{max} = 0.06, 0.07, 0.1, 0.125, 0.15$. The open circles are the frequency shift for a linear probe conducted at $t = 198$; the filled circles are the frequency shift for a linear probe conducted at $t = 202$.

$\Omega_0 = 0.988$ (compare to Figure 1). For $t < 100$, the amplitude $A \approx 0.005$. When the large amplitude drive $F = 0.125$, has been on for a long time the amplitude in the rock $A = 0.394$ (note this is not the linear probe amplitude). Thus the linear probe finds an elastic modulus $K = 1 - \lambda_1 A - \lambda_2 A^2 = 0.648$, and the resonance frequency is $\Omega \approx 0.8$ ($\Omega^2 \approx K$). Immediately after the drive is turned down, $t = 202$, the amplitude relaxes to $A \approx 0.005$, and the resonance frequency shifts to $\Omega \approx 0.89$. The anharmonic contribution to the elastic modulus has decayed away ($\tau_2 = 0$), while the hysteretic contribution remains. Thus K snaps from 0.648 to $K = 1 - \lambda_1 \times 0.394 - \lambda_2 \times (0.005)^2 = 0.803$. The hysteretic component of the elastic modulus relaxes slowly to the low amplitude value, $A \approx 0.005$, and the resonance frequency evolves toward $\Omega_0 = 0.99$.

In Figure 5 we have plotted the resonant frequency shifts as a function of the amplitude for several values of F_{max}. Otherwise, the protocol is identical to that described for Figure 4. The frequency shifts are the difference between $\Omega_0 = 0.99$ and (a) the linear resonant frequency at $t = 198$ (circles), and (b) the linear resonant frequency at $t = 202$ (filled circles). The dependence of the latter frequency shift is linear in the pumping amplitude response and proportional to λ_1. The instantaneous decay in resonant frequency upon changing F from F_{max} to 0.001 is the difference between the open circles and the filled circles. The difference between these two frequencies increases as F_{max} increases since the second order (anharmonic) contribution to the elastic modulus increases. These results are in accord with the preliminary findings of *Tencate* [1997].

Conclusion

In this paper we have introduced a phenomenological model of the dynamic nonlinear elastic response of a rock. The model has two distinct kinds of nonlinearity, a traditional anharmonic contribution, and a contribution derived from elastic hysteresis. Each kind of nonlinearity is allowed to have a slow dynamics. A simple explanation of the experimental findings of *Tencate and Shankland* [1997], in which a slow dynamics makes itself known, is provided by the model.

The existence of a slow dynamics in strain response means that in any experiment that disturbs the rock beyond linearity there will be features of the time history of the disturbance in the response to the disturbance. The enormous separation between time scales (the period of the fundamental is less than 0.01 sec; the time scale for slow dynamics is greater than 100 sec) suggests that many features of the slow dynamics in the nonlinearity may be probed using NMR-style experimental protocols. A preliminary illustration of such a protocol yields results similar to those found in experiment.

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R. A. Guyer, Department of Physics and Astronomy, University of Massachusetts, Amherst, MA 01003. (e-mail: rag@cmp.phast.umass.edu)

K. R. McCall, Department of Physics, University of Nevada, Reno, NV 89557. (e-mail: mccall@physics.unr.edu)

K. Van Den Abeele, EES-4, MS D443, Los Alamos National Laboratory, Los Alamos, NM 87545. (e-mail: vda@lanl.gov)

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