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Observation and implications of nonlinear elastic wave response in rock

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Abstract. Experiments in rock show a large nonlinear elastic wave response, far greater than that of gases, liquids and most other solids. The large response is attributed to structural discontinuities in rock such as microcracks and grain boundaries. The magnitude of the harmonics created by nonlinear interactions grows linearly with propagation distance in one-dimensional systems. In the earth, a large nonlinear response may be responsible for significant spectral alteration of a seismic wave at amplitudes and distances currently considered to be within the linear elastic regime. We argue, based on observations at ultrasonic frequencies, that the effect of nonlinear elasticity on seismic wave propagation may be large, and should be considered in modeling.

Introduction

It is generally assumed that beyond a few source radii, seismic waves propagating outward from the source are in elastically linear material, i.e., in material where stress is linearly proportional to strain. Assuming linear response, recorded spectra from seismic waves are used to estimate the magnitude of the source, characterize high frequency roll-off (important for seismic discrimination purposes) and model source parameters. A nonlinear relationship between stress and strain (nonlinear elasticity) means that frequency components of a wave multiply with each other leading to the creation of sum and difference frequency waves along the propagation path and the failure of wave superposition. We have conducted ultrasonic experiments to study the spectral alteration that takes place along the wave propagation path. Based on our laboratory results we have modeled seismic wave propagation from moderate sized earthquakes in the earth.

Studies of nonlinear behavior in rock show that the nonlinear response in rock is strikingly large, orders of magnitude larger than most solids. One well known manifestation of this behavior is demonstrated by static measurements on rock of velocity or modulus versus applied pressure [*Birch*, 1966]. These tests show a strong nonlinear dependence of modulus on pressure, and therefore a nonlinear stress-strain relation, due to structural discontinuities in the form of cracks, grain boundaries, joints, etc. Nonlinear hysteretic behavior has been seen in measurements of the stress-strain equation of state on rock [*Boitnott*, 1992; *Holcomb*, 1984]. Recent dynamic studies of transient waves in rock at atmospheric pressure demonstrate that rock has a large nonlinear response at relatively small strains [*Zinov'yeva et al.*, 1989; *Ostrovsky*, 1991; *Johnson and Shankland*, 1989]. Nonlinear wave behavior implies that as the wave propagates there is a local increase in the density and modulus during compression and a local decrease in density and modulus during rarefaction. The waveform begins to change shape and sum and difference frequencies are created. The effect is cumulative so that, for example, harmonic amplitudes increase with distance until wave attenuation begins to dominate.

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Background Theory

The equation of motion for a homogeneous elastic solid, to second order in the displacement (cubic anharmonicity in the elastic moduli), is derived in several texts [*Landau and Lifshitz*, 1959; *Murnaghan*, 1951]. The inclusion of attenuation leads to a straightforward modification of this equation [*Polyakova*, 1964; *McCall*, 1993]. For a longitudinal plane wave propagating in the x direction, the equation of motion in the frequency domain and in the absence of attenuation is [*McCall*, 1993]

$$\left(\frac{\partial^2}{\partial x^2}+k^2\right)u(x,\omega)=-S(x,\omega)$$
$$-\beta\frac{\partial}{\partial x}\int\frac{d\omega'}{2\pi}\frac{\partial u(x,\omega')}{\partial x}\frac{\partial u(x,\phi)}{\partial x}, \quad (1)$$

where $u(x,\omega)$ is the displacement, $S(x,\omega)$ is the external source, $k^2=\omega^2/c^2$, c is the compressional velocity, β is the nonlinear coefficient defined as

$$\beta=\frac{3(\lambda+2\mu)+2(\ell+2m)}{2(\lambda+2\mu)}, \quad (2)$$

λ and μ are second-order elastic moduli (Lamé coefficients), and ℓ and m are third-order elastic moduli (Murnaghan coefficients).

The second term on the right-hand side of (1), the nonlinear interaction of the displacement with itself, causes the creation of sum and difference frequency waves. Equation (1) can be solved analytically by an iterative Green function technique [*McCall*, 1993], giving linear and first-order nonlinear displacement terms

$$u_0(x,\omega) = \int dx' g(x,x',\omega) S(x',\omega), \quad (3a)$$

$$u_1(x,\omega) = \beta \int dx' g(x,x',\omega) \int \frac{d\omega'}{2\pi} \cdot \frac{\partial}{\partial x'} \left[\int dx'' \frac{\partial g(x',x'',\omega')}{\partial x'} S(x'',\omega') \cdot \int dx''' \frac{\partial g(x',x''',\phi)}{\partial x'} S(x''',\phi) \right], \quad (3b)$$

where $\phi = \omega - \omega'$ and the Green function $g(x,x',\omega)$ satisfies

$$\left(\frac{\partial^2}{\partial x^2} + k^2 \right) g(x,x',\omega) = -\delta(x - x'). \quad (4)$$

Solution to (1) in an infinite medium for a continuous-wave source at the origin of frequency ω_0 and amplitude U is

$$u(x,t) = u_0 + u_1 = U \sin(k_0|x| - \omega_0 t) - \frac{\beta U^2 k_0^2 x}{4} [1 + \cos(2k_0|x| - 2\omega_0 t)], \quad (5)$$

to first order in the nonlinearity. The source wave of frequency ω_0 interacts with itself, creating waves of frequency $\omega_0 - \omega_0 = 0$ and $\omega_0 + \omega_0 = 2\omega_0$. The nonlinearly generated terms grow linearly with propagation distance x and nonlinear coefficient β, and grow as the square of the source frequency ω_0 and the source amplitude U. If the source contains more than one frequency, the first-order nonlinear term u_1 becomes more complex. The integral over frequency in (3b) means that all source frequencies interact with all other source frequencies. For example, a source composed of two frequencies ω_1 and ω_2 gives rise to waves propagating at frequencies 0, $2\omega_1$, $2\omega_2$, $\omega_1 + \omega_2$, and $\omega_1 - \omega_2$, to first order in the nonlinearity.

Experimental Results

The apparatus used in the experiments is shown in Figure 1. Source displacements ranged from approximately 10^{-9}–10^{-7} m. The most fundamental experimental observation providing evidence for nonlinear elastic behavior in the sample is shown in Figure 2. Figure 2*a* shows the frequency spectrum measured at the source for drive frequencies $\omega/2\pi$ of 7–32 kHz with peak amplitude at approximately 20 kHz. The source is a Blackman window. The seven different curves correspond to progressively increasing drive amplitudes varying over three orders of magnitude. The source displacement spectrum contains only a small fraction of harmonic and intermodulation effects and only at large drive levels. Figure 2*b* shows the resulting displacement frequency spectrum at 1 m. For detected displacements at the fundamental frequency as small as 3×10^{-8} m, the composition of the displacement frequency spectrum at 1 m is extremely rich in frequencies not present at the source. As drive amplitude is increased the spectrum becomes progressively richer due to the nonlinear elastic wave interaction in the material. Harmonic growth is apparent at detector strain levels of less than 10^{-7} in this experiment.

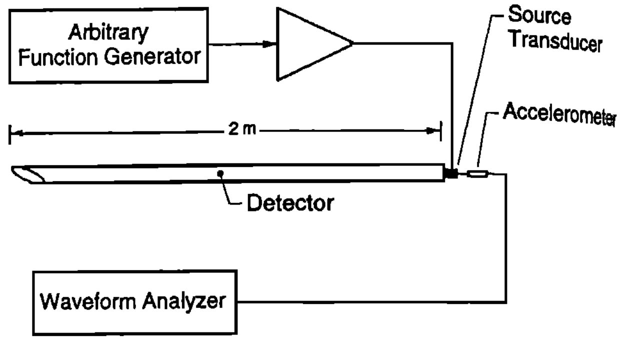

Figure 1. Experimental configuration. The drive transducer is composed of a piezoelectric crystal with a backload. A calibrated accelerometer is attached to the backload for measurement of source amplitudes. From previous experiments we are certain that the displacement measured on the backload is the same as that on the rock surface. The detector is in a small hole drilled into the Berea sandstone rod 1 m from the source transducer. Detected signals are recorded using a 16 bit waveform analyzer. One end of the sample was tapered in order to minimize reflections.

Figure 2*c* shows the theoretical result calculated from (3a) and (3b) at one meter using the Green function of an infinite solid. Both the linear response u_0 (dashed line) and the linear plus first-order nonlinear response $u_0 + u_1$ (long dashed line) are shown. In the model $\beta = -10^3$. There is good qualitative agreement between the model and the experimental results. The shape of the detected experimental spectrum is influenced by the response of the system including the response of transducers and electronics. This is not a large effect, however, and it is linear with drive amplitude. Our work with pure tone drive signals indicates that higher harmonics tend to be strong [*Meegan et al.*, 1993]. The model does not account for this. The presence of harmonics higher than 2ω suggests that higher order terms (i.e., quartic anharmonicity) in the stress-strain relationship are important to a complete description of nonlinear elastic behavior in rock.

The compressional nonlinear parameter β can be calculated from pure-tone source experiments and the relationship between source amplitude and the amplitude at 2ω given by (5). In general, we find that $\beta = -10^3$–-10^4 for the rocks we have studied. Compare this value to liquids such as water or intact metals such as aluminum where $|\beta| < 10$. The value of β for rock is similar to that obtained for a liquid containing gas bubbles [*Hamilton*, 1986; *Bulanov*, 1993].

We have used the simplest form of the theory, (3a) and (3b), to compare with the data, neglecting the effects of attenuation and higher order terms in the solution to (1). We have available the results of a complete treatment of the theoretical problem including attenua-

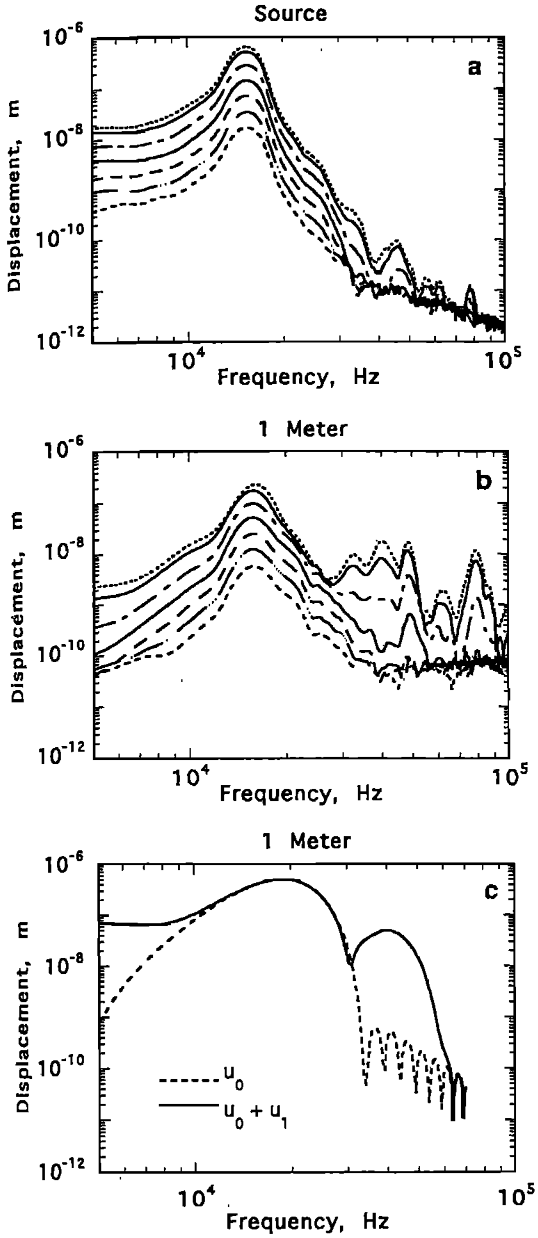

Figure 2. Broadband displacement spectra. The source signal is a Blackman window of peak frequency approximately 20 kHz and band limited at 7 and 32 kHz. (a) The source displacement spectrum at increasing applied voltages is shown by the different line types. (b) Displacement spectrum 1 m from source at increasing applied voltages corresponding to drive levels in Figure 2*a*. Note the richness of the spectrum created by nonlinear elasticity in the rock during wave propagation. (c) Model result showing linear response (dashed line) and nonlinear response (long dashed line) at 1 m for a drive signal identical to that input to the sample. The model result contains no attenuation.

tion [*McCall,* 1993] and a full set of experimental studies of the nonlinear response of the sample. These include experiments conducted with individual pure tones over the entire audio band [*Meegan et al.,* 1993] and experiments conducted using many broad frequency intervals, for example, 0–40 kHz.

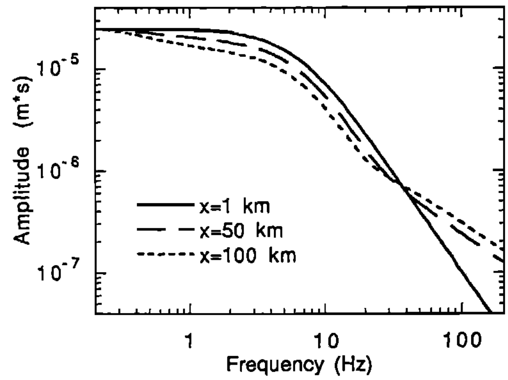

Figure 3. Model displacement spectra at seismic frequencies. The source is broadband [*Yu et al.,* 1992]. The pulse propagates to $x = 1$ km, 10 km and 100 km, producing sum and difference frequencies through a first order nonlinear interaction. The model includes wave attenuation typical for the earths crust in the western U.S. The initial displacement was chosen to correspond to a magnitude 5.5 seismic source.

Discussion

Are strains as low as 10^{-7} large enough to produce a significant nonlinear response at large distances from a seismic source in the earth? We have numerically modeled the propagation of a plane wave produced by a broadband source at seismic frequencies including nonlinear elasticity and attenuation [*McCall,* 1993]. The source function is flat to a corner frequency of approximately 6 Hz and rolls-off as ω^{-2} [*Yu et al.,* 1992]. To be conservative about the possible pressure and frequency dependence of β, we assumed that the earth's crust at seismic frequencies exhibits a nonlinear response an order of magnitude less than that seen in ultrasonic laboratory measurements on rock at atmospheric pressure. We chose $\beta = -200$ and specific dissipation Q (equivalent to energy loss per cycle) for an active tectonic region of 10^2. The initial source displacement amplitude was chosen to be consistent with a magnitude 5.5 source, $U = 10^{-3}$ m [*Denny et al.,* 1987]. In Figure 3, we show the evolution of the displacement frequency spectrum $x = 1$, 50, and 100 km. Clearly, the effect of nonlinear elasticity on the displacement frequency spectrum with distance is significant, especially in the high frequency roll-off portion of the spectrum.

What is the nonlinear region of the crust? Further studies of nonlinear elastic response in rock as a function of pressure, fluid saturation, structural discontinuity, and dimension are necessary in order to address this question. How does nonlinear response depend on frequency? We have observed strong nonlinear response in extensional resonant bar experiments to frequencies below 800 Hz [*Johnson et al.,* 1993]. *Bonner and Wanamaker,* [1991] observe strong nonlinear response in laboratory torsional experiments at seismic frequencies.

The conventional wisdom is that rocks are nonlinear for strains greater than 10^{-6} [*Murphy,* 1982; *Winkler et al.,* 1979]. Quite often nonlinear effects are discussed in terms of a threshold below which nonlinearity does not exist. We argue that the idea of a threshold is an artifact of measurements which attempt to see nonlinearity against a nonzero background, for example, measurements of wave distortion or a shift in resonance frequency. On the other hand, nonlinearly generated harmonics are identically zero unless nonlinear interactions are occurring. The threshold for measuring nonlinear response by studying the growth of harmonics is set by the noise level and linearity of the experimental instrumentation. We suggest that studying the spectral alteration as a function of distance in field data is an excellent way to determine the extent of nonlinear response in the earth. We are currently conducting small-scale explosive tests in the field for this purpose and in the future we will study data from earthquakes and nuclear blasts.

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