Planarization of graphs embedded on surfaces

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Abstract

A planarizing set of a graph is a set of edges or vertices whose removal leaves a planar graph. It is shown that, if G is an n-vertex graph of maximum degree d and orientable genus g, then there exists a planarizing set of $O(\sqrt{dgn})$ edges. This result is tight within a constant factor. Similar results are obtained for planarizing vertex sets and for graphs embedded on nonorientable surfaces. Planarizing edge and vertex sets can be found in $O(n+g)$ time, if an embedding of G on a surface of genus g is given. We also construct an approximation algorithm that finds an $O(\sqrt{gn \log g})$ planarizing vertex set of G in $O(n \log g)$ time if no genus-g embedding is given as an input.

1 Introduction

A graph G is *planar* if G can be drawn in the plane so that no two edges intersect. Planar graphs arise naturally in many applications of graph theory, e.g. in VLSI and circuit design, in network design and analysis, in computer graphics, and is one of the most intensively studied class of graphs [23]. Many problems that are computationally hard for arbitrary graphs have efficient solutions for the case of planar graphs. Examples include minimum spanning trees, network flow problems, shortest path problems, and many others.

If the graph is not planar, then often a problem arises of how to find a planar subgraph that is as close to the given graph, as possible. A problem of this type is called a *graph planarization problem*. This problem has been intensively investigated in relation to its applications to circuit layout [21, 3, 24, 18]. One approach to the graph planarization problem is to construct a maximal planar subgraph of the input graph G. By solving the maximal planar subgraph problem one finds a minimal set of edges whose removal leaves a planar graph (called a *planarizing set* of edges). Several fast algorithms have been recently proposed for the maximal planar subgraph problem, including the $O(m \log n)$ algorithm of Cai et al. [2], the $O(m+n\alpha(m,n))$ algorithm of La Poutré [25] and the $O(m+n)$ algorithm of Djidjev [9] (n and m are the number of vertices and edges respectively). Although a solution of the maximal planar subgraph problem defines a *minimal* planarizing set of

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edges, there is no guarantee that that edge set is of small size. On the other hand the problem of finding a minimum planarizing set of edges is known to be NP-complete [11].

In this paper we show that for any n vertex graph of bounded genus g and maximum degree d there exists a planarizing edge set of size $O(\sqrt{dgn})$. This result is tight within a constant factor and improves the best previous bound of $O(d\sqrt{gn})$ [5]. We also consider the related problem of finding a small planarizing set of *vertices*. Although an asymptotically optimal $O(\sqrt{gn})$ bound for this problem is known [5], the previous proofs are quite complex and the constants are large. We present here very simple proof that also gives a leading constant 4, improving the previous leading constants $44/\sqrt{3}$ of [5, 6] and 26 of [17].

We consider also graphs embedded on nonorientable surfaces, showing that similar bounds hold for the sizes of the smallest edge and vertex planarizing sets.

Our proof technique combines a careful examination of the topology of the graph with a use of a radius reduction device [28]. Our proofs are constructive, giving linear algorithms that find the planarizing sets, if an embedding of the graph on its genus surface is given.

We also investigate the problem of finding a planarizing set in the case where an embedding of the graph on its genus surface is **not** given. This is motivated by the fact that the problem of embedding a graph on a surface of minimum genus is known to be NP-hard [27]. The best known algorithm for the latter problem is polynomial on the number of vertices n, but doubly exponential on the genus g [10]. In this paper we describe an approximation algorithm that finds an $O(\sqrt{dgn\log g})$ planarizing edge set that does not require a genus-g embedding to be given as an input. No comparable algorithm for this problem has been previously constructed.

The paper is organized as follows. Section 2 contains preliminaries. In Section 3 we prove our main result about planarizing edge sets and in Section 4 we prove a similar result about planarizing vertex sets. The nonorientable case is discussed in Section 5. In the last section we describe an efficient algorithm that constructs a planarizing set without knowing the embedding of the graph on its genus surface.

2 Preliminaries

By a *surface*, we mean a closed connected 2-manifold [14, 12]. There are two major types of surfaces: orientable and nonorientable surfaces. Informally, if an intelligent bug starts from some point on a closed curve drawn on a surface, traverses the curve, and returns to the initial point with its initial orientation reversed, then the curve is *orientation reversing*. Otherwise, the curve is *orientation preserving* [14]. A surface is *nonorientable* if there is at least one orientation reversing closed curve on the surface. Otherwise, it is *orientable*. Excluding the sphere, the simplest orientable surface is the torus, and the simplest nonorientable surface is the projective plane. Every orientable (nonorientable) surface is the sum of a certain number g of tori (projective planes), and this number $g \geq 0$ is the *genus* of the surface [14]. By *orientable genus surface* of a graph, we mean the orientable surface of minimum genus on which the graph can be embedded. By *orientable genus* of a graph, we mean the genus of its orientable genus surface. The terms *nonorientable genus surface* and *nonorientable genus* are similarly defined. A *2-cell* is a region that is homeomorphic to the open unit disc in R^2.

An important relation between the number of vertices, edges, and faces of any 2-cell embedding of a graph on a surface is given by the Euler's formula [12, 14].

Theorem 1 *: Let an embedding of an n-vertex, m-edge f-face graph G on a surface S of genus g be given such that every one of the faces of the embedding is a 2-cell. Then the equality $n-m+f = X(S)$ holds, where $X(S) = 2-2g$, if S is orientable, or $X(S) = 2-g$, if S is nonorientable.*

The number $X(S)$ from Theorem 1 is called the *Euler characteristic* of S.

A simple closed curve c is called *contractible* if c divides S into two disjoint regions one of which is a 2-cell. Otherwise c is *noncontractible.* If c is noncontractible curve, then c is either *noncontractible separating*, if c divides S into two disjoint regions (none of which is a 2-cell), or *noncontractible nonseparating* otherwise. Similarly, if G is a graph embedded on S, then any simple cycle of G will be called respectively contractible, noncontractible separating and noncontractible nonseparating depending on the corresponding curve on S.

Let G be a connected graph with spanning tree T of radius r and root t. Each nontree edge of G forms with the edges of T a unique simple cycle. Call each such cycle a *fundamental* cycle. The length of any fundamental cycle is at most $2r+1$, if it contains t, or at most $2r-1$, otherwise.

We are going to use the following known topological facts (for simple combinatorial proofs see, e.g. [7, 13].)

Lemma 2.1 *Let c be a noncontractible nonseparating curve on a surface S. Then $S-c$ can be embedded on a surface of characteristic $X(S)+2$, if c is orientation preserving, or $X(S)+1$, if c is orientation reversing.*

Lemma 2.2 *Consider a 2-cell embedding of a graph G with a spanning tree of radius r and a root t on an arbitrary surface S. Then there exists a set of $2-X(S)$ fundamental cycles with a total of no more than $2r(2-X(S))+1$ vertices, one of which is t, whose deletion from G leaves behind a planar graph.*

3 Planarizing sets for graphs embedded on orientable surfaces

Let G be an n-vertex connected graph of orientable genus g. (If G is not connected we apply the same argument on any connected component of G of positive genus.)

In order to apply Lemma 2.2 we need to reduce (in an appropriate sense) the radius of the graph (to $O(\sqrt{n/g})$ for planarizing vertex sets and $O(\sqrt{m/(4dg)})$ for edge sets). Since the construction for vertex sets is simpler, we will consider this case first. Then we will prove our main result for planarizing edge sets. We begin this section by proving two key lemmas.

3.1 The radius reducing device

Embed G on an orientable surface S of genus g and add new edges in order to obtain a triangulation. Divide the vertices of G into levels according to their distance from some fixed vertex of G.

The idea of our procedures constructing vertex and edge planarizing sets is the following. Firstly, we find a small set of vertices (resp. edges) whose removal divides G into components each containing $O(\sqrt{n/g})$ (resp. $O(\sqrt{m/(4dg)})$) levels. Secondly, we add a small number of new edges resulting in a graph of genus $O(g)$ and radius $O(\sqrt{n/g})$ (resp. $O(\sqrt{m/(4dg)})$). Finally, we apply Lemma 2.2. For implementing the second step we need the following two lemmas.

For some constant r to be defined below, denote by L_j, for $1 \leq j \leq r$, the set of vertices on all levels i such that $i \bmod r = j$. If the graph has at least r levels, then there exists a $i' \leq r$ such that $|L_{i'}| \leq \lfloor n/r \rfloor$. Let $M = L_{i'},\ \ l_1 < \cdots < l_s$ be the set of vertices in levels in M, and E_M be the set of all edges (v, w) of G such that $w \in M$ and the level of v is lower than the level of w.

Lemma 3.1 *Let d be the number of components induced by the vertices of M and let the deletion of E_M from G results in a graph of k' components and genus g'. Then $g' \leq g - d + k' - 1$.*

Proof: Let $i \leq s$ and L be any component induced by the set of vertices at level l_i. Consider the subgraph L' of L induced by the set of the vertices of L adjacent to vertices at level $l_i + 1$. Then L' is a union of edge disjoint simple cycles [22] implying that L contains at least one simple cycle. Denote by c one such simple cycle. Delete all edges of E_M incident to vertices of c. Let G' be the resulting graph and k_c and g_c be the number of components and the genus of G'. If c is a contractible or a noncontractible separating cycle, then $k_c \geq k + 1$, $g_c \leq g$. If c is a noncontractible nonseparating cycle, then $k_c \geq k$, $g_c \leq g - 1$, where k is the number of components of G. In both cases $k_c - g_c \geq k - g + 1$. Delete all remaining edges of E_M incident to vertices of L. Let k_L and g_L be the number of components and the genus of the resulting graph. Then $k_L \geq k_c$ and $g_L \leq g_c$. From the previous inequality

$$k_L - g_L \geq k - g + 1. \tag{1}$$

Applying this construction on all connected components L of the subgraph of G induced by M, we obtain a k'-component graph of genus g'. By induction from (1) $k' - g' \geq k - g + d$ and since by assumption G is connected and $k = 1$ then $g' \leq g - d + k' - 1$. ∎

Notice that in the proof of Lemma 3.1 we constructed an embedding of $G - E_M$ on an orientable surface S' such that

$$X = X(S') \geq X(S) + 2d. \tag{2}$$

For each level $l > 0$ from $L_{i'}$, delete all edges connecting a vertex on level $l - 1$ with a vertex on level l. Let k' and g' denote the number of components and the genus of the resulting graph G' and let K be any component of G'. Let l^* be the lowest level of K and K^* be the subgraph of K induced by the set of vertices on level l^*. Add to K a minimum number, e_K, of edges to make K^* connected. This increases the genus by no more than e_K. Let d_K denote the number of the components of K^*. Then

$$e_K \leq d_K - 1. \tag{3}$$

Obviously, if $l^* > 0$ then $l^* \in L_{i'}$. Shrink the subgraph induced by the vertices on level l^* in K (which after the addition of the new edges is connected) to a point t^*. The resulting component has a spanning tree rooted at t^* and radius not exceeding $r-1$. Repeat the same procedure with all other components of G' and denote by G'' the resulting graph.

Lemma 3.2 *Each component of G'' has a spanning tree of radius not exceeding $r-1$ and the genus of G'' is less or equal than g.*

Proof: Denote by d the sums of d_K over all components K of G' and by g'' the genus of G''. By Lemma 2.2, for each nonplanar component K'' of G'' a planarizing vertex set $P_{K''}$ of no more than $4g_{K''}(r-1)+1$ vertices of G'' exists, where $g_{K''}$ is the genus of K''. However, we will show that at most $4g_{K''}(r-1)$ of these vertices belong to G. If $P_{K''}$ contains a vertex corresponding to a shrunk subgraph induced by $L_{i'}$, then $|P_{K''}| \leq 4g_{K''}(r-1)$. Otherwise $P_{K''}$ will not contain the root of the spanning tree of the corresponding component of G'' and thus $|P_{K''}| \leq 4g_{K''}(r-2)+1$. Since the genus of any graph is equal to sum of the genera of its connected components [1], then g'' is equal to the sum of $g_{K''}$ over all components K'' of G''. From this fact and (3) applied to all components K of G' we have $g'' - g' \leq d - k' + 1$, and by Lemma 3.1 $g' + d - k' + 1 \leq g$.

By combining the last two inequalities $g'' \leq g$ follows. ∎

3.2 Finding a planarizing vertex set

The following theorem improves the best previous bound by a factor of 6.5.

Theorem 2 *Let G be an n-vertex graph of orientable genus g. Then there exists a set of no more than $4\sqrt{gn}$ vertices of G whose removal leaves a planar graph.*

Proof: Suppose that G is connected. Embed G on a surface of genus g and add an appropriate number of edges to triangulate the embedding. Divide the vertices of G into levels according to their distance from some vertex t. Let $r = \lceil \sqrt{n/(4g)} \rceil$. Denote by L_j for $1 \leq j \leq r$ the set of vertices on all levels i such that $i \bmod r = j$. If there are no more than $r-1$ levels in G, then Lemma 2.2 directly implies the existence of a $4\sqrt{gn}$ planarizing vertex set. Assume then that the number of levels is at least r. Find a level $i' \leq r$ as in the previous subsection such that $|L_{i'}| \leq \lfloor n/r \rfloor$ and for each level $l > 0$ from $L_{i'}$ delete all edges connecting a vertex on level $l-1$ with a vertex on level l. Modify the resulting graph by adding new edges and shrinking the vertices on the lowest level of each component so that (by Lemma 3.2) the resulting graph has genus at most g and each component has radius at most $r-1$. Then a planarizing vertex set for G exists that includes $L_{i'}$ and contains no more than $\lfloor n/r \rfloor + 4g(r-1) \leq n/\lceil \sqrt{n/(4g)} \rceil + 4g\lfloor \sqrt{n/(4g)} \rfloor \leq 4\sqrt{gn}$ vertices.

If G is not connected, then for each component K of G there exists a planarizing vertex set of no more than $4\sqrt{n_K g_K}$ vertices, where n_K and g_K are the number of vertices and genus of K, respectively. Since $\sum_K n_K = n$ and $\sum_K g_K = g$, then $\sum_K 4\sqrt{n_K g_K} \leq 4\sqrt{ng}$, which proves the theorem in the general case. ∎

3.3 Finding a planarizing edge set

The *degree* of a graph denote the maximum degree of its vertices.

Theorem 3 *For any m-edge graph of degree d and orientable genus g, there exists a planarizing set of no more than $4\sqrt{dgm}$ edges.*

Proof: Embed the given graph on some orientable surface S of genus g. Make each face a triangle by adding a suitable number of edges and call the resulting graph G.

To obtain a small planarizing edge set we shall define now $L_{i'}$ to be an edge set, instead of a vertex set as in Theorem 2. This is done as follows. In the graph G, assign a weight of 0 to all new triangulating edges and a weight of 1 to all m edges of the original graph. Divide the vertices into levels according to their distance from some vertex t. Choose $r = \lceil\sqrt{m'/(4dg)}\rceil$ where m' is the total weight of all edges connecting vertices at different levels. Denote by L_j for $0 \leq j < r$ the set of all edges between level i and level $i+1$ for all i such that $i \bmod r = j$. Assume again that the number of levels is at least r. Then there exists an $i' < r$ such that the weight of $L_{i'}$ is no more than $\lfloor m'/r \rfloor$. Therefore, $L_{i'}$ contains no more than $\lfloor m'/r \rfloor$ original edges.

Delete $L_{i'}$ from G, and add a number of edges to obtain a graph G' of genus not exceeding g and radius of each component not exceeding $r-1$ as in Lemma 3.2. As shown in the proof of Theorem 2, a planarizing vertex set of G' of no more than $4g(r-1)$ vertices can be found. Choosing the edges incident with these vertices, we obtain a planarizing edge set of G' of size at most $4gd(r-1)$. Then a planarizing edge set for G exists of size not exceeding

$$\lfloor m'/r \rfloor + 4dg(r-1) \leq m'/\sqrt{m'/(4dg)} + 4dg\sqrt{m'/(4dg)} = 4\sqrt{dgm'} \leq 4\sqrt{dgm}. \quad \blacksquare$$

Next we will give an estimation of the maximum size of a planarizing edge set of G, given the number of vertices of G (instead of the number of edges as in Theorem 3).

Lemma 3.3 *Consider a triangulation embedding of G with f faces on an arbitrary orientable surface. Let the vertices of G be divided into levels according to their distance from some vertex t. Let U_t^G be the set of all the edges that connect two vertices at the same level in G. Then $|U_t^G| \geq f/2$.*

Proof: Consider any face (x_1, x_2, x_3) of the triangulation. It is not possible that $level(x_i) - level(x_j) > 1$ for any $0 \leq i, j \leq 2$. Therefore, at least one of the three edges of each face must belong to U_t^G. Since each edge is adjacent to only two faces, the result follows. $\blacksquare$

Theorem 4 *Let G be a connected n-vertex graph of degree d and orientable genus g. There exists a set of no more than $4\sqrt{2dg(n+2g-2)}$ edges whose removal makes G planar.*

Proof: Consider without a loss of generality a triangulation embedding of G on its genus surface. Then, by Lemma 3.3, $|U_t^G| \geq f/2 = m/3 = n + 2g - 2$. Furthermore, by the proof of Theorem 3, there exists a planarizing edge set of size not exceeding $4\sqrt{dgm'}$ for G. Then

$$4\sqrt{dgm'} = 4\sqrt{dg(m - |U_t^G|)} \leq 4\sqrt{2dg(n+2g-2)}. \quad \blacksquare$$

The *skewness* of a graph is defined in [15] as the smallest number of edges whose removal leads to a planar graph. Then by Theorem 4 the skewness of the class of n-vertex g-genus d-degree graphs is not exceeding $4\sqrt{2dg(n+2g-2)}$.

4 Extension to the Nonorientable Case

We show here that results from the previous sections can be extended to graphs embedded on nonorientable surfaces. As we will show, we can use similar arguments as in the orientable case, with a few modifications. We will only sketch the main steps of the proof.

Let G be a connected graph. Consider a triangulation embedding of G on a nonorientable surface S of genus g. Divide the vertices of G into levels according to their distance to some vertex t. Denote by r the maximum level and let M be the set of all vertices on certain nonadjacent levels $0 < l_1 < \ldots < l_j \leq r$. Let E_M be the set of all edges (v, w) of G such that $w \in M$ and the level of v is lower than the level of w. Then we shall prove the following analogue of Lemma 3.1.

Lemma 4.1 *If d is the number of components induced by the vertices of M and if the deletion of E_M turns G into a k'-component graph G', then the components of G' can be suitably distributed to form a graph of orientable genus g^* and a graph of nonorientable genus g^{**} such that $2g^* + g^{**} \leq g - 2d + 2k' - 2$.*

Proof: The construction from the proof of Lemma 3.1 can be applied to G and S in order to obtain a graph G' embedded in k surfaces whose sum X of characteristics satisfies (2). Moreover, by construction each of these k' surfaces embeds a component of $G - E_M$. Thus $X \leq 2k - 2g^* - g^{**}$ and from (2) $2 - g + 2d \leq 2k' - 2g^* - g^{**}$, and thus $2g^* + g^{**} \leq g - 2d + 2k' - 2$. ∎

Theorem 5 *If G is an n-vertex graph of degree d and nonorientable genus g, then there exists an edge set of size $2\sqrt{2dgm}$ whose removal from G leaves a planar graph.*

Proof: Consider a triangulation embedding of G on its nonorientable genus surface S. Define m' as in the proof of Theorem 3. Choose $r = \lceil\sqrt{m'/(2dg)}\rceil$ and find the set $L_{i'}$ as in Theorem 3. Assume that the number of levels is at least $2r$ (otherwise the theorem follows directly from Lemma 2.2). As in the proof of Theorem 3, delete all edges connecting a vertex on level $l-1$ with a vertex on level l for each level $l > 0$ from $L_{i'}$. Let k denote the number of components of the resulting graph G'. Add a number of edges so that each component of the resulting graph G'' has a radius not exceeding $r-1$. Furthermore, by Lemma 4.1, the components of G'' form two subgraphs of G', one of orientable genus g^* and the other of nonorientable genus g^{**}, such that $2g^* + g^{**} \leq g$. Then, applying Lemma 2.2 on each nonplanar component of G'', we find a planarizing edge set of size no more than

$$\lfloor m'/r \rfloor + 4dg^*(r-1) + 2dg^{**}(r-1) \leq \lfloor m'/r \rfloor + 2dg(r-1)$$
$$\leq m'/\sqrt{m'/(2dg)} + 2dg\sqrt{m'/(2dg)} = 2\sqrt{2dgn}. \blacksquare$$

For planarizing vertex sets we have the following theorem.

Theorem 6 *If G is an n-vertex graph of nonorientable genus g, then there exists a vertex set of size $2\sqrt{2gn}$ whose removal from G leaves a planar graph.*

5 Tightness of the results

The tightness in the orientable case follows from the fact that $\Omega(\sqrt{gn})$ vertices or $\Omega(\sqrt{dgm})$ edges are needed in the worst case to separate an n-vertex m-edge graph of degree d and genus g [7, 13, 26]. If one can planarize a graph by removing $o(\sqrt{gn})$ vertices or $o(\sqrt{dgm})$ edges, then one can then find in the resulting planar graph a separator of $O(\sqrt{n})$ vertices [20] or $O(\sqrt{dn})$ edges [4], which will be a contradiction to the above lower bounds.

For graphs embedded on nonorientable surfaces the tightness of our results follows from the tightness in the orientable case. Let G be any n-vertex graph of orientable genus g. Consider a 2-cell embedding of G on an orientable surface of genus g. By adding a projective plane to S we obtain an embedding of G on a nonorientable surface of Euler characteristic $X(S) - 1 = 2 - 2g - 1$. Thus, by Theorem 1, the nonorientable genus of G is at most $2g + 1$.

6 Algorithmic aspects

The constructive proofs of the theorems from the previous sections can be transformed into linear algorithms that find planarizing sets satisfying the theorems, if an embedding of the graph on its orientable (resp. nonorientable) genus surfaces is known. Finding such an embedding, however, is a very hard computational problem itself. In this section we show that almost optimal planarizing vertex set (upto a factor of $O(\sqrt{\log g})$) can be found for any n-vertex g-genus graph in $O(n \log g)$ time. We will make use of the following result from [8].

Theorem 7 *For any n vertex graph G a partitioning A, B, C of the vertices of G can be found in $O(n)$ time such that no edge joins a vertex in A with a vertex in B, $|A|,|B| \leq n/2$, $|C| \leq c\sqrt{(g'+1)n}$, and the graph induced by the set of vertices $A \cup B$ has genus not exceeding $g - g'$, where g is the genus of G, $0 \leq g' \leq g$, and c is a global constant.*

The following algorithm finds a planarizing vertex set without knowing an embedding of the graph on its genus surface.

Algorithm PLANARIZE

Input: A non-planar graph G.

1. Initialize: $C = \emptyset$, $M = \{G\}$, where M denotes a set of nonplanar edge disjoint subgraphs of $G - C$.

2. Perform Steps $3 \div 6$ until $M = \emptyset$.

3. Pick any graph G_i from M and and update $M := M - \{G_i\}$.

4. Divide G_i as in Theorem 7 into two disconnected subgraphs with no more than $n_i/2$ vertices each, by removing a set C_i of $O(\sqrt{(g_i+1)n_i})$ vertices of G_i, where g_i and n_i are the genus and the number of vertices of G_i.

5. Use the algorithm from [16] to test each subgraph for planarity.

6. Let $C := C \cup C_i$. Add the nonplanar subgraphs found in Step 5 to M.

Let us estimate the maximum time $T(n,g)$ and the maximum size $S(n,g)$ of the set C found by Algorithm PLANARIZE for any n-vertex g-genus graph G. We have the relations
$T(n,g) \le kn + max\{T(n/2,g_1) + T(n/2,g_2) | g_1 + g_2 \le g\}$, if $n > 0,\ g > 0,$
$T(n,0) = O(1), \quad T(0,g) = O(1)$
and
$S(n,g) \le \max\{c\sqrt{n(g_3+1)} + S(n/2,g_1) + S(n/2,g_2) | g_1 + g_2 + g_3 = g, g_3 \ge 0,\ g_1, g_2 > 0\}.$
$S(n,0) = O(1), \quad S(0,g) = O(1).$

Solving the recurrences we find $T(n,g) = O(n\log(2g))$ and $S(n,g) = O(\sqrt{gn\log g})$. Thus we have the following theorem.

Theorem 8 *For any n-vertex graph G a planarizing vertex set of G of size $O(\sqrt{gn\log g})$ can be found in $O(n\log(2g))$ time, where $g > 0$ is the orientable genus of G, without being given an embedding of G on its genus surface.*

A similar result holds for planarizing sets of edges.

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