The Schwinger–Dyson equations and non-renormalization in Chern–Simons theory

D. Daniel and N. Dorey
Physics Department, University of Edinburgh, Mayfield Road, Edinburgh EH9 3JZ, Scotland, UK

Received 25 April 1990

The Schwinger–Dyson equations for the propagators of the Chern–Simons–Witten model are analysed using Ward identities implied by the recently discovered $N=1$ supersymmetry of the gauge-fixed action. An integral constraint on the wave-function renormalization constant is derived and shown to imply the vanishing of all radiative corrections to the ghost and gauge propagators in the Landau gauge.

1. Introduction

The finiteness of the Chern–Simons–Witten (CSW) [1] model is an essential ingredient in its relation to the invariants of knots and links, and to conformal field theory. The calculation of ref. [2] indicates not only that the model is finite up to two loops in perturbation theory but also that all radiative corrections to two- and three-point functions vanish to this order. Recently Birmingham et al. [3,4] have discovered an additional (super)symmetry of the Landau gauge Chern–Simons action which can be used to constrain the Lorentz tensor structure of the gauge-field propagator, and it has been speculated [4,5] that the finiteness properties of the model have their origin in this new invariance. In this letter we will show that the Ward identities of this symmetry actually force the kernels of the Schwinger–Dyson equations for the gauge and ghost propagators to vanish, implying that the exact, non-perturbative, two-point functions are simply the bare ones.

This approach assumes that the supersymmetry is unbroken at the quantum level but in fact recent work [6] has shown that if the theory is regulated in a fully gauge invariant way then anomalous finite contributions occur at one loop which are equivalent to the finite shift in the coupling constant originally observed in ref. [1]. However, it is shown in an accompanying letter [7] that although a gauge invariant regularization scheme breaks the symmetry, Ward identities for the broken symmetry still hold and in fact reproduce the shift in the coupling found in ref. [6]. Hence the aim of the present analysis is simply to demonstrate that in the absence of corrections to the Ward identities there are no radiative corrections to the gauge or ghost propagators.

2. The Chern–Simons action and its symmetries

The Chern–Simons action in three-dimensional euclidean space is given by

$$S_{CS}=\frac{k}{4\pi}\int d^3x\,\epsilon^{\mu\nu\rho}(\tfrac{1}{2}A^a_\mu\,\partial_\nu A^a_\rho+(1/3!)f^{abc}A^a_\mu A^b_\nu A^c_\rho), \tag{1}$$

where f^{abc} are the structure constants of a semi-simple Lie algebra and k is a positive integer. As in four-dimensional Yang–Mills theory, path-integral quantization requires integration over gauge-inequivalent

 0370-2693/90/$ 03.50 © 1990 – Elsevier Science Publishers B.V. (North-Holland)

configurations only, necessitating the introduction of gauge-fixing and ghost terms by the usual Fadeev–Popov construction,

$$S_{\rm GF}=\frac{k}{4\pi}\int d^3x\, A^a_\mu\, \partial^\mu B^a,\quad S_{\rm FP}=\int d^3x(\partial_\mu \bar{c}^a)(D^\mu c)^a, \tag{2, 3}$$

where $(D_\mu c)^a=\partial_\mu c^a+f^{abc}A^b_\mu c^c$ and B is a Lagrange multiplier field enforcing the Landau gauge condition [2].

In addition to the usual BRS symmetry, the full gauge-fixed action, $S=S_{\rm CS}+S_{\rm GF}+S_{\rm FP}$, has a further global invariance [3,4] under the following transformations:

$$\delta A^a_\mu=\epsilon^\beta\epsilon_{\mu\beta\gamma}\,\partial^\gamma c^a,\quad \delta c^a=0,\quad \delta\bar{c}^a=\frac{k}{4\pi}\epsilon^\beta A^a_\beta,\quad \delta B^a=\epsilon^\beta(D_\beta c)^a, \tag{4}$$

where ϵ^β is a constant, anticommuting vector parameter (hence this is referred to as an $N=1$ supersymmetry). Following ref. [4], the invariance of the path-integral under these transformations leads to Ward identities relating the Greens functions of the theory. In terms of the effective action Γ,

$$\int d^2x\left(\epsilon_{\mu\beta\gamma}\frac{\delta\Gamma}{\delta A^a_\mu}\partial^\gamma c^a+\frac{k}{4\pi}A^a_\beta\frac{\delta\Gamma}{\delta\bar{c}^a}+\frac{\delta\Gamma}{\delta B^a}\frac{\delta\Gamma}{\delta S^a_\beta}\right)=0, \tag{5}$$

where S_β is a source for the non-linear variation of the multiplier field B. Functionally differentiating this identity with respect to A^ν and c and setting the fields equal to zero gives a relation between the two-point gauge and ghost functions

$$\frac{\delta^2\Gamma[0]}{\delta A^a_\mu(x)\delta A^b_\nu(y)}=\frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\frac{\partial^x_\lambda}{\partial^2_x}\frac{\delta^2\Gamma[0]}{\delta\bar{c}^a(x)\delta c^b(y)}, \tag{6}$$

where the transversality of the Landau gauge propagator has been imposed. In fact it can be shown that this Ward identity holds for the transverse part of the gauge propagator in arbitrary covariant gauge.

When gauge invariant regulator terms, such as Pauli–Villars fields and higher derivative interactions [6], are added to the action they break the supersymmetry and lead to corrections to this Ward identity. However it can be shown [7] that the effect of these corrections is to reproduce the shift in the coupling $k\to k+c_{\rm v}$, where $c_{\rm v}$ is the quadratic Casimir in the adjoint representation of the gauge group, previously observed in refs. [1,6]. In what follows we neglect this effect.

One additional differentiation of (5) with respect to A^ρ prior to setting the fields equal to zero yields a relationship between the three-point functions of the theory. In momentum space, using the equation of motion for the B field, this Ward identity is

$$\epsilon_{\mu\beta\gamma}(ik^\gamma)\Gamma^{\mu\nu\rho}_{abc}(k,p,q)=\frac{k}{4\pi}[\delta^\rho_\beta\Gamma^\nu_{bca}(p,q,k)+\delta^\nu_\beta\Gamma^\rho_{cba}(q,p,k)-iK^\rho_{\beta bca}(p,q,k)p^\nu-iK^\nu_{\beta cba}(q,p,k)q^\rho], \tag{7}$$

where $\Gamma^{\mu\nu\rho}$ is the 1PI three-point gauge function and Γ^ν is the irreducible $(A^\nu,\bar{c},c)$-vertex. The auxiliary function K^α_β is given by

$$K^\alpha_{\beta abc}(p,q,k)=\int d^3x\, d^3y\, d^3z\, \exp[-i(p\cdot x+q\cdot y+k\cdot z)]\frac{\delta^3\Gamma[0]}{\delta S^{\beta a}(x)\delta A^b_\alpha(y)\delta c^c(z)}. \tag{8}$$

3. The Schwinger–Dyson equations

The Schwinger–Dyson equations are an infinite hierarchy of integral equations between the Greens functions of a quantum field theory which usually require drastic approximations before any solutions can be obtained. Remarkably, in the CSW model the Ward identities of supersymmetry are sufficiently restrictive to decouple the equations for the two-point gauge and ghost functions from the others and allow an exact treatment. The

Schwinger–Dyson equations are derived from the path integral in the usual way by considering the integral of a functional derivative of the action with respect to the fields (see, for example, ref. [8]). The form of the full gauge fixed action (1)–(3) ensures that these equations are similar diagrammatically to their counter-parts in QCD, with the obvious omission of terms coming from the extra vertices of the QCD lagrangian (four-gluon, etc., ...). The equation for the ghost propagator, $S_{ab}(p)=S(p)\delta_{ab}$, is (see fig. 1)

$$S^{-1}_{ab}(p)=S^{-1}_{(0)}(p)\delta_{ab}-\mathrm{i}p^{\mu}f^{acd}\int\frac{\mathrm{d}^3k}{(2\pi)^3}S(k)\Delta_{\mu\nu}(q)\Gamma^{\nu}_{bcd}(q,k,-p). \tag{9}$$

where $q=p-k$ and $\Delta^{\mu\nu}$ is the full gauge propagator. Eq. (9) is an exact relation between unrenormalized Greens functions which are potentially divergent and must be made finite. The theory will be regulated by introducing some cut-off, Λ, in all loop integrals. This procedure is not gauge-invariant and it is tacitly assumed that gauge invariance is recovered when the cut-off is removed at the end of the calculation, i.e. that there is no gauge anomaly. The nature of the Schwinger–Dyson equations makes the implementation of gauge-invariant regularization schemes extremely difficult.

The euclidean space Feynman rules derived from the action (1)–(3) give the bare ghost and gauge propagators

$$S^{ab}_{(0)}(p)=\frac{\delta^{ab}}{p^2},\quad \Delta^{(0)ab}_{\mu\nu}(p)=\delta^{ab}\frac{4\pi}{k}\epsilon_{\mu\nu\lambda}\frac{(-\mathrm{i}p)^{\lambda}}{p^2}. \tag{10, 11}$$

If the exact ghost propagator is written in terms of the bare one as $S(p)=H(p^2)S_{(0)}(p)$ where H is an unknown wave-function renormalization function, the Ward identity (6) constrains the exact gauge propagator;

$$\Delta^{ab}_{\mu\nu}(p)=\delta^{ab}\frac{4\pi}{k}H(p^2)\epsilon_{\mu\nu\lambda}\frac{(-\mathrm{i}p)^{\lambda}}{p^2}. \tag{12}$$

The main result of this calculation will be to demonstrate that $H(p^2)$ is identically equal to one. In order to simplify the Schwinger–Dyson equation for the ghost propagator it is necessary to expand the gauge-ghost-ghost vertex as a three-vector in terms of the obvious basis;

$$\Gamma^{\mu}_{abc}(p,k)=\mathrm{i}[A_{abc}(p,k)p^{\mu}+B_{abc}(p,k)k^{\mu}+C_{abc}(p,k)\epsilon^{\mu\alpha\beta}p_{\alpha}k_{\beta}], \tag{13}$$

where p and k are incoming momenta on the anti-ghost and ghost legs respectively. The Feynman rule for the bare vertex, Γ^{μ} corresponds to $A=-f^{abc}$ and $B=C=0$ and it has been shown [2] that this receives no perturbative corrections up to two loops. However, higher order or even non-perturbative contributions cannot be ruled out.

Substituting the expressions (12), (13) for $\Delta_{\mu\nu}$ and Γ^{ν} in eq. (9) leads to considerable simplification. The antisymmetry of the $\epsilon^{\mu\nu\lambda}$ tensor causes the terms involving the functions $A(p,k)$ and $B(p,k)$ to vanish. The structure constants f^{acd} in (9) project out the part of the vertex which is totally antisymmetric in group indices. Writing $C^{abc}(p,k)$ as $f^{abc}C(p,k)+$symmetric parts, and using the identity $f^{acd}f^{bcd}=c_{\mathrm{v}}\delta^{ab}$ gives a relation between the functions $H(p^2)$ and $C(p,k)$

$$\frac{1}{H(p^2)}=1-\frac{4\pi c_{\mathrm{v}}\mathrm{i}}{k}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{H(k^2)H(q^2)}{p^2k^2q^2}C(k,-p)[p^2k^2-(p\cdot k)^2]. \tag{14}$$

The other Schwinger–Dyson equation to be considered here is the equation for the gauge propagator in terms of the other two- and three-point functions (see fig. 2).

$$\Delta^{\mu\nu}_{ab}(p)^{-1}=\Delta^{\mu\nu}_{(0)ab}(p)^{-1}-[\mathscr{L}^{\mu\nu}_{ab}(p)+\mathscr{M}^{\mu\nu}_{ab}(p)] \tag{15}$$

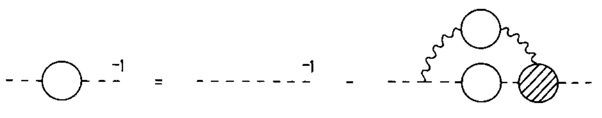

Fig. 1. The Schwinger–Dyson equation for the ghost propagator.

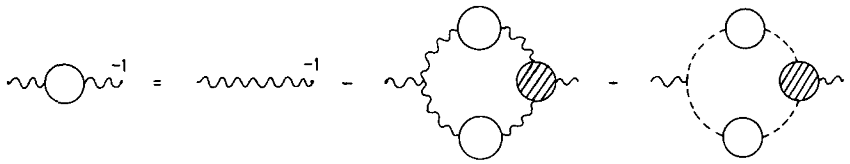

Fig. 2. The Schwinger-Dyson equation for the gauge propagator.

where $\mathscr{L}^{\mu\nu}$ and $\mathscr{M}^{\mu\nu}$ are the kernels coming from gauge field self-interaction and gauge-ghost interactions respectively.

$$\mathscr{L}^{\mu\nu}_{ab}(p)=\frac{k}{4\pi}f^{acd}\epsilon^{\mu\beta\alpha}(\tfrac{1}{2})\int\frac{d^3k}{(2\pi)^3}\Delta_{\alpha\gamma}(k)\Delta_{\beta\delta}(q)\Gamma^{\gamma\delta\nu}_{cdb}(k,q,-p), \tag{16}$$

$$\mathscr{M}^{\mu\nu}_{ab}(p)=(-1)\int\frac{d^3k}{(2\pi)^3}(iq^\mu f^{adc})S(k)S(q)\Gamma^{\nu}_{bcd}(-p,k,q). \tag{17}$$

Substituting for $\Delta^{\mu\nu}$ from (12) in (15) and inverting the antisymmetric tensor gives

$$\frac{\delta^{ab}}{H(p^2)}=\delta^{ab}-\frac{2\pi i}{k}\epsilon_{\mu\nu\rho}\frac{p^\rho}{p^2}[\mathscr{L}^{\mu\nu}_{ab}(p)+\mathscr{M}^{\mu\nu}_{ab}(p)]. \tag{18}$$

The first term on the RHS of (18) involves the contraction $\epsilon_{\mu\nu\rho}p^\rho\Gamma^{\gamma\delta\nu}_{cdb}(k,q,-p)$ which can be expressed in terms of ghost functions using the Ward identity (7). Substituting everywhere for $\Delta^{\mu\nu}$ and Γ^μ using (12), (13) and performing the tensor algebra, making repeated use of the identity $\epsilon^{\mu\alpha\beta}\epsilon^{\mu\delta\gamma}=\delta^{\alpha\delta}\delta^{\beta\gamma}-\delta^{\alpha\gamma}\delta^{\beta\delta}$, it is found that all terms involving the unknown tensor K^α_β (coming from the RHS of (7)) cancel amongst each other and, as in the ghost equation, all terms involving the functions $A(p,k)$ and $B(p,k)$ vanish. Similarly the group tensor structure projects out the vertex part proportional to f^{abc} giving another integral equation relating the functions $H(p^2)$ and $C(p,k)$.

$$\frac{1}{H(p^2)}=1-\frac{4\pi c_v i}{k}\int\frac{d^3k}{(2\pi)^3}\frac{H(k^2)H(q^2)}{p^2k^2q^2}[C(k,-p)-\tfrac{1}{2}C(k,q)][p^2k^2-(p\cdot k)^2]. \tag{19}$$

Comparison with (14) reveals

$$\int\frac{d^3k}{(2\pi)^3}\frac{H(k^2)H(q^2)}{p^2k^2q^2}C(k,q)[p^2k^2-(p\cdot k)^2]=0. \tag{20}$$

Multiplying (14) by $H(p^2)$ and integrating with respect to p gives

$$\int d^3p[H(p^2)-1]=\frac{4\pi c_v i}{k}\int d^3p\int\frac{d^3k}{(2\pi)^3}\frac{H(p^2)H(k^2)H(q^2)}{p^2k^2q^2}C(k,-p)[p^2k^2-(p\cdot k)^2]. \tag{21}$$

However, performing the same operation to (20) gives

$$\int d^3p\int\frac{d^3k}{(2\pi)^3}\frac{H(p^2)H(k^2)H(q^2)}{p^2k^2q^2}C(k,q)[p^2k^2-(p\cdot k)^2]=0. \tag{22}$$

This integral is well defined when the cut-off is removed and so an interchange in the order of integration and a linear shift in integration variables are allowed. In particular the change of variables $p\to k-p$ shows that this integral is equal to the one on the RHS of (21). Thus, allowing Λ to become infinite in (21) and performing the trivial angular integration on the LHS, gives

$$\int_0^\infty p\,dp^2[H(p^2)-1]=0. \tag{23}$$

The above relation is an integral constraint on the continuum wave-function renormalization, in particular it rules out any momentum-independent divergences in $H(p^2, \Lambda^2)$ as $\Lambda \to \infty$. Thus no renormalization of the bare coupling k is required at the level of two-point functions. This is consistent with the expectation that k should be held equal to a fixed positive integer as $\Lambda \to \infty$ in order to preserve gauge invariance [1]. The vanishing of the beta function has also been demonstrated by Blasi and Collina [9] as a consequence of the unbroken scale invariance of the theory at the quantum level. In fact scale invariance dictates that the dimensionless function $H(p^2)$ can only be a constant, in which case (23) clearly implies that

$$H(p^2) \equiv 1, \tag{24}$$

or, equivalently, that the exact ghost and gauge propagators are just the free ones.

4. Conclusion

This calculation has demonstrated that, up to anomalous contributions to the Ward identities of supersymmetry, two-point functions in the CSW model are not renormalized in the Landau gauge. It seems likely that similar methods may be applied successfully to higher Greens functions. In fact (5) generates an infinite hierarchy of Ward identities relating the n-point functions of the theory to each other. When combined with the Schwinger–Dyson equations which relate n-point functions to $(n+1)$-point functions and separated into parts of different tensor structure they may be sufficiently restrictive to prove non-renormalization for general n.

Supersymmetry frequently improves the UV divergences of Greens functions in quantum field theory. In most higher dimensional cases (SQCD, SUGRA, etc., ...) it is implemented by introducing superpartners for the existing fields in the action. Cancellations then occur between graphs and non-renormalization theorems can often be proven to all orders in perturbation theory. The CSW model is an interesting example of a case where a supersymmetry between the existing degrees of freedom (gauge and ghost) leads to finiteness, and the Schwinger–Dyson equations of the theory may be treated exactly to prove a non-renormalization theorem for two-point functions in a non-perturbative context.

Acknowledgement

We would like to thank Nick Brown, Peter Higgs, Richard Kenway, Tien Kieu and Brian Pendleton for numerous helpful discussions. This work is partly supported by the States of Guernsey Education Council.

References

[1] E. Witten, Commun. Math. Phys. 121 (1989) 351.
[2] E. Guadagnini, M. Martellini and M. Mintchev, Perturbative aspects of the Chern–Simons field theory, CERN preprint CERN-TH.5324/89.
[3] D. Birmingham, M. Rakowski and G. Thompson, Nucl. Phys. B 329 (1990) 83.
[4] D. Birmingham and M. Rakowski, Superfield formulation of Chern–Simons supersymmetry, ICTP preprint ICTP-IC/89/38.
[5] F. Delduc, F. Gieres and S.P. Sorella, Supersymmetry of the $D=3$ Chern–Simons action in the Landau gauge, CERN preprint CERN-TH.5391/89.
[6] L. Alvarez-Gaumé, J.M.F. Labastida and A.V. Ramallo, A note on perturbative Chern–Simons theory, CERN preprint CERN-TH.5480/89.
[7] N. Dorey, Phys. Lett. B 246 (1990) 87..
[8] C. Itzykson and J.B. Zuber, Quantum field theory (McGraw-Hill, New York, 1980) p. 475.
[9] A. Blasi and R. Collina, Finiteness of Chern–Simons model in perturbation theory, Genova preprint GEF-Th-12/1989.