Symmetry Breaking with a Slant: Topological Defects after an Inhomogeneous Quench

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We show that in second-order phase transformations that are induced by an inhomogeneous quench the density of topological defects is drastically suppressed as the velocity with which the quench propagates falls below a threshold velocity. This threshold is approximately given by the ratio of the healing length to relaxation time at freeze-out, which is the instant when the critical slowing down results in a transition from the adiabatic to the impulse behavior of the order parameter. [S0031-9007(99)09414-4]

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In a uniform cosmological phase transition characterized by an order parameter with a nontrivial homotopy group, creation of topological defects is virtually inevitable. As pointed out by Kibble [1] some years ago, different domains shall select different low-temperature vacua, leading to irreconcilable differences which condense out into topological defects. Thus, special relativity implies disorder in the context of the big bang, yielding a useful lower bound on the initial density of defects. An analogous situation is also encountered in the condensed matter setting [2], suggesting the possibility of experimental investigation of a cosmological scenario. For this case, however, a lower bound based on the light-cone causal independence is no longer useful. In condensed matter systems (and most likely also in the cosmological second-order phase transitions), the initial density of defects has to be computed through arguments which rely on the dynamics of the order parameter [2,3].

The estimate of defect density proposed by one of us relies on the observation that, as a result of critical slowing down, the state of the system which crosses the critical region at a finite pace will inevitably cease to keep up with the changes of thermodynamic parameters at some point sufficiently near the critical temperature [2,3]. In a homogeneous quench, this will happen everywhere at the instant when the characteristic time $\epsilon/\dot{\epsilon}$, at which changes in the mass term of the effective Landau-Ginzburg (LG) potential $V(\phi) = [\phi^4 - 2\epsilon(t)\phi^2]/8$ become faster than the relaxation time, i.e., $\tau = \tau_0|\epsilon|^{-1}$ in the overdamped case. When $\epsilon(t) \simeq t/\tau_Q$, this leads to a freeze-out time $\hat{t} \simeq (\tau_Q\tau_0)^{1/2}$. At $t = -\hat{t}$ the order parameter ϕ becomes too sluggish to keep up with the evolving shape of the effective potential. An evolution dominated by dynamics driven by $V(\phi)$ shall resume $\hat{t}$ after the phase transition, which takes place when $\epsilon(t=0) = 0$. The interval $[-\hat{t}, +\hat{t}]$ represents the impulse stage, namely, the period during which the effect of $\partial V/\partial\phi$ is negligible, although fluctuations and damping continue to matter. At $\hat{t}$, $\hat{\epsilon} \equiv \epsilon(\hat{t}) = (\tau_0/\tau_Q)^{1/2}$, which leads to the estimate of the corresponding healing (or correlation) length $\hat{\xi} = \xi_0(\tau_Q/\tau_0)^{1/4}$. This freeze-out healing length constrains the domain size relevant for defect formation. The corresponding relaxation time is $\hat{\tau} = (\tau_Q\tau_0)^{1/2}$, and the characteristic phase ordering speed is $\hat{v} = \hat{\xi}/\hat{\tau} = v_0(\tau_0/\tau_Q)^{1/4}$, where $v_0 = \xi_0/\tau_0$. This overdamped LG example can be generalized [3].

Homogeneous quenches are a convenient idealization and may be a good approximation in some cases. However, in reality, the change of thermodynamic parameters is unlikely to be ideally uniform. Thus, the mass parameter $\epsilon(t, \vec{r})$, varying in both time and space, must be considered in defect formation. As a consequence, locations entering the broken symmetry phase first could then communicate their choice of the new vacuum as the phase ordered region spreads in the wake of the phase transition front. When inhomogeneity dominates, symmetry breaking in various, even distant, locations is no longer causally independent. The domain where the phase transition occurred first may impose its choice on the rest of the volume, thus suppressing or even halting production of topological defects.

Experiments carried out in ^{3}He [4,5], where a small volume of superfluid is reheated to normal state, and subsequently rapidly cools to the temperature of the surroundings, are a good example of an inhomogeneous quench: The normal region shrinks from the outside. Yet, topological defects are created, thus suggesting that the phases of distinct domains within the reheated region are selected independently. Even in ^{4}He, where the transition can be induced by a change of pressure, it is difficult to rule out the possibility that a quench may be somewhat nonuniform, thus causing decrease of the density of defects, which could explain the recent evidence of nonappearance of vortices [6].

Here we consider two idealized cases of inhomogeneous quenches: (i) a *shock wave* in which $\epsilon(t, \vec{r})$ is a propagating step in space and (ii) a *linear front* in which $\epsilon(t, \vec{r})$ linearly interpolates between the pre- and post-transition values. For both scenarios, we investigate the threshold velocity v_t at which the phase ordered region expands behind the $\epsilon = 0$ critical point. This velocity is given by the ratio of the healing length to relaxation time set by the dynamics. In an inhomogeneous quench, v_t will be—as it was anticipated early on [2]—of the order of $\hat{v}$.

In an insightful paper directly stimulated by the ^{3}He experiments, Kibble and Volovik have argued that the initial density of defects should conform with the homogeneous quench estimates of [2,3] when the velocity v of the inhomogeneous quench front exceeds $\hat{v}$; on the other hand, the initial density of defects should be suppressed as $v/\hat{v}$ for the case of a slowly spreading phase transition [7]. We find that $\hat{v}$ indeed plays a crucial role. However, our study shows that the suppression is even more dramatic. When $\hat{v} > v$, essentially no defects appear.

Shock wave.—To begin with, we consider a decay of a false symmetric vacuum to a true symmetry broken phase in a one-dimensional dissipative ϕ^4 model. In dimensionless units,

$$\ddot{\phi} + \eta\dot{\phi} - \phi'' + \frac{1}{2}(\phi^3 - \epsilon_0\phi) = 0\,, \qquad (1)$$

where $\phi(t,x)$ is a real order parameter and ϵ_0 is a positive constant. We look for a solution $\phi(t,x)$ which interpolates between $\phi(t,-\infty) = -\sqrt{\epsilon_0}$ and $\phi(t,+\infty) = 0$. Such a solution cannot be static; it is a stationary half-kink

$$\phi(t,x) = -\sqrt{\epsilon_0}\left(1 + \exp\left[\frac{\sqrt{\epsilon_0}\,(x - v_t^s t)}{2[1-(v_t^s)^2]^{1/2}}\right]\right)^{-1} \qquad (2)$$

moving with characteristic velocity

$$v_t^s = \left[1 + \left(\frac{2\eta}{3\sqrt{\epsilon_0}}\right)^2\right]^{-1/2} \overset{\eta\to\infty}{\approx} \frac{3\sqrt{\epsilon_0}}{2\eta}\,. \qquad (3)$$

Our shock wave quench is a sharp pressure front propagating with velocity v. That is,

$$\ddot{\phi} + \eta\dot{\phi} - \phi'' + \frac{1}{2}[\phi^3 - \epsilon(t,x)\phi] = \vartheta(t,x)\,, \qquad (4)$$

where

$$\epsilon(t,x) = \epsilon_0\,\mathrm{sgn}(t - x/v) \qquad (5)$$

is the space-time dependent effective mass and $\vartheta(t,x)$ is a Gaussian white noise of temperature θ with correlations

$$\langle\vartheta(t,x)\vartheta(t',x')\rangle = 2\eta\theta\delta(t-t')\delta(x-x')\,. \qquad (6)$$

There is a unique vacuum ($\phi = 0$) to the right of the propagating front ($x > vt$), and the symmetry is broken ($\phi = \pm\sqrt{\epsilon_0}$) behind the front ($x < vt$).

The field in the $\phi = 0$ vacuum does not respond instantaneously to the passing front. There are two qualitatively different regimes:

(1) $v > v_t^s$: The phase front propagates faster than the false vacuum can decay. The half-kink (2) lags behind the front (5); a supercooled symmetric phase grows with velocity $v - v_t^s$. The supercooled phase cannot last for long; it is unstable, and the noise ϑ makes it decay into one of the true vacua. Since the noise does not have any bias, the decay results in production of kinks.

(2) $v < v_t^s$: The phase front is slow enough for a half-kink to move in step with the front, $\phi(t,x) = H_v(x - vt)$. The symmetric vacuum decays into one definite nonsymmetric vacuum, the choice is determined by the boundary condition at $x \to -\infty$. No topological defects are produced in this regime. To make sure that the field cannot flip, we must check if the stationary solution $H_v(x - vt)$ is stable against small perturbations.

We investigate the stability in the $\eta \to \infty$ limit when the system is overdamped and the $\ddot{\phi}$ term in Eq. (4) can be neglected. H_v is most likely to be unstable for $v = v_t^{s-}$. We use the ansatz

$$\phi(t,x) = H_{v_t^s}(x - v_t^s t) + f(x - v_t^s t)\exp\left(\Gamma t - \frac{\eta v_t^s x}{2}\right). \qquad (7)$$

The eigenequation turns out to be time independent;

$$-\Gamma\eta f(x) = -f''(x) + \left[\frac{\epsilon_0}{2}\,\mathrm{sgn}(x) + \frac{\eta^2(v_t^s)^2}{4} + \frac{3}{2}H_{v_t^s}^2(x)\right]f(x)\,. \qquad (8)$$

The "potential" in the square brackets is positive definite for $v_t^s = 3\sqrt{\epsilon_0}/2\eta$. This proves the stability of $H_{v_t^s}$ at sufficiently low noise temperature θ.

In the opposite $\eta \to 0$ limit the half-kink (2) just below the threshold $v_t^s \approx 1$ becomes a step function $H_{v_t^s}(x - v_t^s t) \approx \sqrt{\epsilon_0}[-1 + \mathrm{sgn}\,(x - v_t^s t)]/2$. The potential on the right-hand side of Eq. (8) is again positive for any x.

In summary, no topological defects are produced for $v < v_t^s$. At $v = v_t^s$, there is a sudden jump in the density of defects left behind the shock wave. As v increases above v_t^s, the density saturates at a value characteristic for an instantaneous uniform quench with $\epsilon(t,x) = \epsilon_0\,\mathrm{sgn}(t)$. With increasing θ, the discontinuity at $v = v_t^s$ will be softened. For $v \gg v_t^s$, where the quench is effectively homogeneous, the density of defects will grow logarithmically with θ [8].

These expectations are borne out by the numerical study of kink formation which uses the same code as in Ref. [9]. We illustrate them in Fig. 1 for $\eta = \epsilon_0 = 1$. For all but the highest temperature $\theta = 0.1$, there are essentially no kinks produced in quenches with the velocity of less than 0.8, which is in good agreement with our analytic estimate $v_t^s = 0.83$. However, for the highest temperature, defects appear at a subthreshold velocity. We note that at this temperature potential barrier separating the two minima of the LG potential is comparable with θ.

Linear front.—Let us consider now the linear inhomogeneous quench

$$\ddot{\phi} + \eta\dot{\phi} - \phi'' + \frac{1}{2}[\phi^3 - \epsilon(t,x)\phi] = \vartheta(t,x)\,,$$

$$\epsilon(t,x) = \begin{cases} -\epsilon_0, & \epsilon_0 v\tau_Q \le x - vt\,, \\ \frac{vt-x}{v\tau_Q}, & -\epsilon_0 v\tau_Q \le x - vt \le \epsilon_0 v\tau_Q\,, \\ \epsilon_0, & x - vt \le -\epsilon_0 v\tau_Q\,. \end{cases} \qquad (9)$$

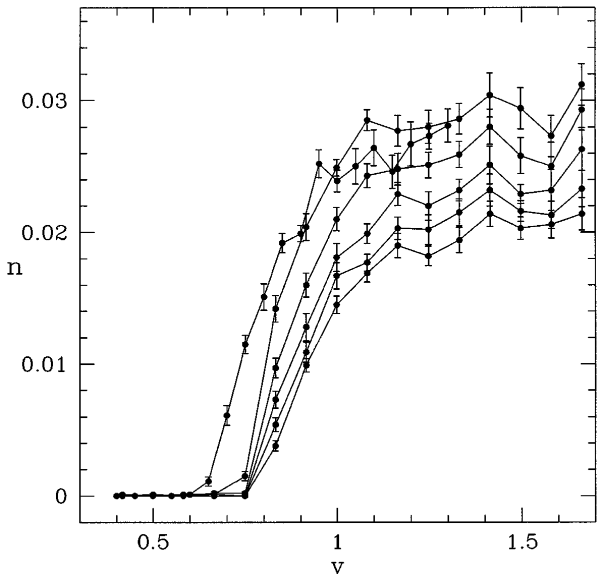

FIG. 1. Density of kinks n as a function of velocity v for the shock wave (5) with $\eta = \epsilon_0 = 1$ (overdamped system). In this overdamped regime, the predicted threshold velocity is $v_t^s = 0.83$. The plots from top to bottom correspond to $\theta = 10^{-1}$, 10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}, and 10^{-10}. At low θ, we get a clear cutoff velocity at $v \approx 0.8$, which is consistent with the prediction.

We assume that the linear part of $\epsilon(t,x)$, namely, the interval $-\epsilon_0 v\tau_Q \leq x - vt \leq \epsilon_0 v\tau_Q$, is much wider than the healing length to the left of the front, $2\epsilon_0 v\tau_Q \gg 1/\sqrt{\epsilon_0}$. If not then the shock limit (5) applies.

In the absence of noise, the propagating linear front is followed by a stationary half-kink $\phi(t,x) = h_v(x - vt)$. This half-kink lags a distance δx behind the front. δx can be estimated by similar arguments as those which led to Eq. (3). At δx behind the front $x = vt$ the mass parameter is $\epsilon_{\delta x} = \delta x/v\tau_Q$. The replacement $\epsilon_0 \rightarrow \epsilon_{\delta x}$ in Eq. (3) gives a velocity $v_t(\delta x)$ the half-kink would propagate with if it were at δx. The velocity increases with δx. The half-kink gets stuck at such a δx that this velocity is equal to the actual front velocity v, $v_t(\delta x) = v$. This takes place at $\delta x = 16\eta^4 v^5 \tau_Q/81(1 - v^2)^2$, which grows quickly with v.

When v is greater than v_t^s in Eq. (3), $\delta x > \epsilon_0 v\tau_Q$ and the half-kink does not stay in the linear regime. It enters the $\epsilon = \epsilon_0$ area where it moves forward with velocity $v_t^s < v$. Like in the shock limit the supercooled phase grows at a constant rate and decays giving rise to kinks.

When $v < v_t^s$, the half-kink remains confined in the linear regime. Even in this case, for v greater than certain threshold v_t, the width δx of the supercooled phase may be large enough for this phase to be unstable. A half-kink $h_v(x - vt)$ confined to the linear regime satisfies

$$(1 - v^2)h_v''(x) + \eta v h_v'(x) - \frac{x}{2v\tau_Q} h_v(x) - \frac{1}{2} h_v^3(x) = 0 . \tag{10}$$

We rescale $x = c_1\tilde{x}$ and $h_v = c_2\tilde{h}$ in Eq. (10), then set $c_1 = [2v\tau_Q(1 - v^2)]^{1/3}$ and $c_2 = \sqrt{c_1/v\tau_Q}$ to obtain the rescaled equation

$$L_1[\tilde{h}](\tilde{x}) \equiv \tilde{h}'' + p\tilde{h}' - \tilde{x}\tilde{h} - \tilde{h}^3 = 0 , \tag{11}$$

with primes now denoting derivatives with respect to $\tilde{x}$. Equation (11) has a single parameter

$$p = \frac{2^{1/3}\eta v^{4/3}\tau_Q^{1/3}}{(1 - v^2)^{2/3}} . \tag{12}$$

The half-kink h becomes unstable at a threshold $p = p_t$. At this critical p_t, $\tilde{h}$ has a zero mode $\tilde{\delta h}$, which satisfies

$$L_2[\tilde{\delta h}](\tilde{x}) \equiv \tilde{\delta h}'' + p_t\tilde{\delta h}' - \tilde{x}\tilde{\delta h} - 3\tilde{h}^2\tilde{\delta h} = 0 . \tag{13}$$

The value of p_t was found in two steps. First, we found solutions to Eq. (11) for a range of p with the relaxation scheme $\dot{\tilde{h}} = L_1[\tilde{h}]$. We applied then the relaxation scheme $\dot{\tilde{\delta h}} = L_2[\tilde{\delta h}]$ to the linear Eq. (13) with the initial condition $\tilde{\delta h}(t = 0, x) = 1$. The field relaxed to $\tilde{\delta h}(t \rightarrow \infty, x) = 0$ for $p < p_t$, and it blew up without limit for $p > p_t$. For $p \approx p_t$, we observed a long lived localized zero mode structure. The threshold estimated in this way is $p_t = 6.5$–6.6.

Defects can be produced for $v > v_t$, where

$$v_t = \left(1 + \frac{2^{1/2}\eta^{3/2}\tau_Q^{1/2}}{p_t^{3/2}}\right)^{-1/2} \overset{\eta\to\infty}{\approx} \frac{p_t^{3/4}}{\eta}\left(\frac{\eta}{\tau_Q}\right)^{1/4} . \tag{14}$$

The instability appears because the eigenvalue of the lowest mode of a linearized fluctuation operator around h_v passes through zero when $v = v_t$. The passage is smooth, so we do not expect any discontinuity in the density of defects as a function of v. For the same reason, we expect the threshold at v_t to be gradually softened with increasing noise temperature θ. For $v \gg v_t$, the inhomogeneity of the quench is irrelevant, and the density of defects can be estimated by scaling argument [2,3] as for a homogeneous quench with a time scale τ_Q.

This analysis is confirmed by the numerical study of linear quenches shown in Fig. 2. For the lowest temperatures, there are no kinks formed below the threshold, which for our $\eta = 1$ is $v_t \approx 0.77$. However, as temperature increases from $\theta = 10^{-10}$ to $\theta = 0.1$, kinks begin to appear at velocities as low as ~ 0.42. This decrease of the threshold for kink formation is now more gradual than for the shock wave case of Fig. 1.

Concluding remarks.—We found that for both the shock and the linear inhomogeneous quench, the density of defects is drastically suppressed for quench velocities lower than the characteristic velocity $v_t \sim \hat{v}$. This prediction was verified by numerical simulations for kinks in one dimension. The theory can be generalized to higher dimensions and to complex order parameter in a straightforward manner.

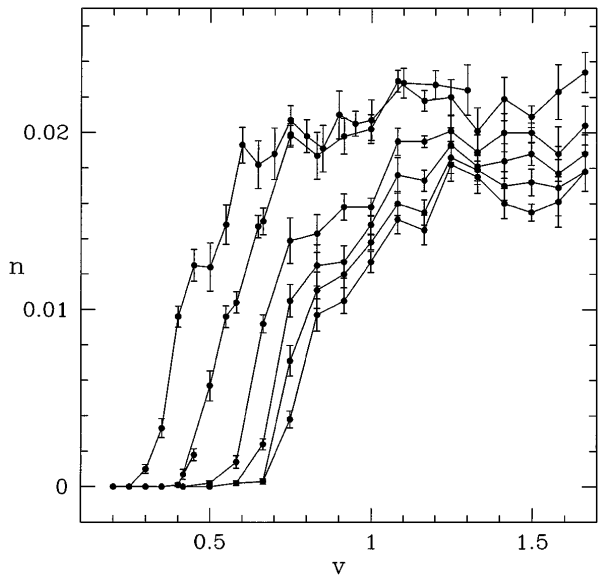

FIG. 2. Density of kinks n as a function of velocity v for the linear inhomogeneous quench (9) with $\tau_Q = 64$ and $\eta = 1$. The predicted threshold is $v_t = 0.77$. This cutoff is achieved for low θ. The plots from top to bottom correspond to $\theta = 10^{-1}$, 10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}, and 10^{-10}.

More quantitative understanding of the dependence of the number of defects on the quench velocity requires further investigation. The non-Hermitian nature of the operator $L_2[\cdots]$ in Eq. (13) should be carefully taken into account.

Our prediction that no defects are produced below certain threshold is in contrast with Ref. [7], where some defects are predicted even at very low velocities (but see Ref. [10] for a different conclusion). Formation of planar solitons in ^{3}He-A [11], which takes place at velocities 10^2–10^3 times lower than the threshold velocity, seems to confirm the latter theory. Let us, however, make the following two remarks.

(1) No uniform temperature gradient was deliberately applied in this experiment although some nonuniformities could result in local temperature gradients. There may have existed such regions of the sample where the gradient was null or negligible. Solitons could be easily created in these areas. This explanation can be verified in the same experimental setup; application of a sufficiently strong uniform temperature gradient across the sample should suppress any soliton formation.

(2) Moreover, in three dimensions, the situation is slightly more complicated. Straightforward generalization of our analysis would rule out formation of zero-dimensional defects, as well as one and especially two dimensional defects parallel or askew to the direction of the quench front. However, such extended string and domain-wall-like structures could still "grow" in the direction perpendicular to the front (and parallel to the direction of its propagation), providing that their seeds exist, say, along the wall where the symmetry is broken first. Strings would grow from some seeds and antistrings from the other. The growth of individual (anti-)strings would not be perfectly perpendicular to the front; they would be wandering around in a chaotic manner. From time to time the growing ends of a string and an antistring would meet and coalesce to form a "jumping rope" with its other ends anchored at the original seeds. For global strings (like vortices in ^{4}He) the coalescences are accelerated by a long range mutual string-antistring attraction. Such a rope would shrink to the original wall thus disappearing from the bulk. At some stage all possible coalescences would have already taken place leaving only a net number of, say, strings, proportional to the square root of the number of seeds. Some of these survivors would be forced by their mutual repulsion to terminate on the side walls. Only a fraction of them would reach the opposite wall spanning through the bulk where they can be unambiguously detected. Their density is likely to be, in any case, orders of magnitude below the estimates based on [2,3]. A similar argument applies to membranelike solitons in [11].

We think that inhomogeneity is a factor which may also need to be taken into account in interpretation of the recent negative ^{4}He experiment [6].

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