Environment-Assisted Invariance, Entanglement, and Probabilities in Quantum Physics

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I introduce environment-assisted invariance or *envariance*—a symmetry exhibited by correlated quantum systems and related to causality—and describe how it can be used to understand the nature of ignorance and, hence, the origin and interpretation of Born's rule for quantum probabilities.

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Quantum theory has a peculiar feature conspicuously absent from classical physics: One can know precisely the state of a composite object (consisting, for example, of the system $\mathcal{S}$ and the environment $\mathcal{E}$) and yet be ignorant of the state of $\mathcal{S}$ alone. The purpose of this paper is to introduce environment-assisted invariance, or *envariance*, to capture this counterintuitive quantum symmetry that allows an observer to use his perfect knowledge (of $\mathcal{SE}$) as a proof of his ignorance of $\mathcal{S}$: When a $u_\mathcal{S}$ acting on $\mathcal{S}$ alone can be undone by a transformation acting solely on $\mathcal{E}$, so that the joint state of $\mathcal{SE}$ is unchanged, this state will be said to be envariant with respect to $u_\mathcal{S}$.

Clearly, envariant properties do not belong to $\mathcal{S}$ alone. Hence, entanglement between $\mathcal{S}$ and $\mathcal{E}$ that enables envariance implies ignorance about $\mathcal{S}$. Envariance is associated with phases of the Schmidt decomposition of the state of $\mathcal{SE}$. It anticipates some of the consequences of environment-induced superselection ("einselection") and allows one to derive and interpret Born's rule [1] relating amplitudes and probabilities, in a manner more physically motivated than the theorem of Gleason [2].

It has become increasingly popular to associate the transition from quantum to classical with decoherence [3–5] and its key consequence, einselection of pointer states [6,7]. Pointer states remain unperturbed in spite of immersion of the system in the environment. This allows for predictability and other symptoms of "objective existence," cornerstones of classicality. However, while this approach has had notable successes, its very foundation is sometimes regarded as *ad hoc*, opening it to a charge of providing the solution "for all practical purposes only" [8]. In particular, as it was pointed out by supporters and detractors alike [9–11], the relation of quantum states to probabilities is *not* settled by decoherence: Born's rule has to be postulated separately. Yet, it is used to arrive at the concept of the reduced density matrix [12,13]—the key tool of the decoherence program.

To motivate envariance, we imagine a system $\mathcal{S}$ entangled with a dynamically decoupled environment $\mathcal{E}$:

$$|\psi_{\mathcal{SE}}\rangle = \sum_{k=1}^{N} \alpha_k |\sigma_k\rangle|\varepsilon_k\rangle. \tag{1}$$

The question we now pose is: Given the state of the combined $\mathcal{SE}$ expressed in the Schmidt form—with complex α_k and with $\{|\sigma_k\rangle\}$ and $\{|\varepsilon_k\rangle\}$ orthonormal—what sort of invariant *quantum facts* can be known about $\mathcal{S}$?

The usual answer would be to use $\psi_{\mathcal{SE}}$ to obtain a reduced density matrix of the system:

$$\rho_\mathcal{S} = Tr_\mathcal{E}|\psi_{\mathcal{SE}}\rangle\langle\psi_{\mathcal{SE}}| = \sum_{k=1}^{N} |\alpha_k|^2 |\sigma_k\rangle\langle\sigma_k|. \tag{2}$$

This step presumes Born's rule [$p_k = |\alpha_k|^2$ is employed [12,13] to get from Eq. (1) to (2)]. Therefore, we cannot take it: We are looking for a more fundamental reason to *trace out the environment*, and we aim to *derive* Born's rule. If successful, such derivation would in turn justify tracing, reduced density matrices, etc.

In order to proceed, we can rely only on these principles of quantum theory that manifestly do not employ Born's rule. To this end, we identify properties of the entangled state $\psi_{\mathcal{SE}}$ that do not belong to $\mathcal{S}$ alone. The strategy is straightforward: Apply transformations that act on the Hilbert space $\mathcal{H}_\mathcal{S}$ of the system and investigate whether their effect *on the joint state* $\psi_{\mathcal{SE}}$ can be undone by "countertransformations" acting solely on $\mathcal{H}_\mathcal{E}$. When the transformed property of the system can be so "untransformed" by acting only on the environment, it is not the property of $\mathcal{S}$. Hence, when $\mathcal{SE}$ is in the state $|\psi_{\mathcal{SE}}\rangle$ with this characteristic, it follows that the envariant properties of $\mathcal{S}$ must be completely unknown.

This motivating discussion leads to the more formal definition of envariance: When for a certain $|\psi_{\mathcal{SE}}\rangle$ and for $U_\mathcal{S} = u_\mathcal{S} \otimes \mathbf{1}_\mathcal{E}$ there exists a $U_\mathcal{E} = \mathbf{1}_\mathcal{S} \otimes u_\mathcal{E}$ such that

$$U_\mathcal{E}(U_\mathcal{S}|\psi_{\mathcal{SE}}\rangle) = |\psi_{\mathcal{SE}}\rangle, \tag{3}$$

then for this state the properties of $\mathcal{S}$ affected—transformed—by $u_\mathcal{S}$ (and, in particular, connected with any observables that do not commute with $u_\mathcal{S}$) are *envariant.*

To paraphrase Bohr's famous dictum about quantum theory, "if the reader does not find envariance strange, he has not understood it": A state of two coins (say, a penny and a cent) would be envariant when the first penny could be flipped, and then the cent "counterflipped," so that the joint state is restored, i.e., no measurement on $\mathcal{SE}$ reveals any difference from the initial state. Pure (perfectly known) quantum states can exhibit this symmetry. If this was the case classically, one would conclude that the observer who could not tell the difference caused by the coin flips must have been in part ignorant about their initial state (that is, he could have known the correlation between coins — e.g., only that they were "same side up"). This connection between envariance and ignorance anticipates our approach to Born's rule.

In quantum theory, envariance is possible for pure joint states. This is because of entanglement. Assume that the joint state $|\psi_{S\mathcal{E}}\rangle$ is pure, expressed in Schmidt form, Eq. (1). Pairs of u_S and $u_\mathcal{E}$ that satisfy $u_S \otimes u_\mathcal{E}|\psi_{S\mathcal{E}}\rangle = \sum_{k=1}^N \alpha_k(u_S|\sigma_k\rangle)(u_\mathcal{E}|\varepsilon_k\rangle) = \sum_{k=1}^N \alpha_k|\sigma_k\rangle|\varepsilon_k\rangle$ exist for an arbitrary set of coefficients α_k: Such u_S are generated by the Hamiltonians of the system that have Schmidt eigenstates $\{|\sigma_k\rangle\}$. For, in this case, the only effect on the system is the rotation of the phases of the coefficients in the Schmidt decomposition:

$$u_S|\sigma_k\rangle = e^{i\omega_k^S t_S}|\sigma_k\rangle = e^{i\varphi_k}|\sigma_k\rangle. \tag{4}$$

Any such u_S can be countered by a $u_\mathcal{E}$:

$$u_\mathcal{E}|\varepsilon_k\rangle = e^{-i\omega_k^\mathcal{E} t_\mathcal{E}}|\varepsilon_k\rangle = e^{-i(\varphi_k + 2\pi l_k)}|\varepsilon_k\rangle, \tag{5}$$

where l_k is an integer. Phases of the coefficients in the Schmidt decomposition can be arbitrarily changed by local interactions. Note that we are affecting phases of the coefficients solely by acting on the states. Note also that — in this case — eigenvalues of the system Hamiltonian $\{\omega_k^S\}$ can be selected at random. It is the matching of $e^{i\omega_k^S t_S}$ with $e^{-i\omega_k^\mathcal{E} t_\mathcal{E}}$ (allowing for the obvious freedom of choice of the eigenvalues, etc.) that matters.

We conclude that an envariant description of the system must ignore phases of the coefficients in Eq. (1). Such description must be based on a set of pairs $\{|\alpha_k|, |\sigma_k\rangle\}$. Hence, something with the information content of the reduced density matrix (i.e., an object dependent solely on $|\alpha_k|$ and on the associated states) provides a complete description of S alone given that the overall state of $S\mathcal{E}$ has a form we have assumed. This conclusion assures *causality*: Phases of Schmidt coefficients of $\psi_{S\mathcal{E}}$ can be influenced by acting on $\mathcal{E}$ alone. If this could be detected by measuring S, faster than light communication would be possible. Indeed, we could have used causality to argue independence of the state of S from any operations (including phase rotations) carried out on $\mathcal{E}$. The assumption of causality is, however, more "costly" (and not entirely quantum) as compared to envariance.

To justify Born's rule, we still need the relation between $|\alpha_k|$ and probabilities. On the other hand, from the uniqueness of Schmidt decomposition we have already recovered the set of the preferred states $\{|\sigma_k\rangle\}$: They are the "fixed points," eigenstates of the envariant transformations u_S (as well as of the local Hamiltonians that generate such u_S). In a sense, this is yet another derivation of the pointer states [6]. Schmidt states are known to enjoy an intimate relationship with them, and have been even regarded as "instantaneous pointer states" [14,15]. One can think of such properties invariant under envariant transformations as *quantum facts.*

The set $\{|\sigma_k\rangle\}$ is not unique when the absolute values of a subset of the coefficients $|\alpha_k|$ are equal and nonzero. I now turn to investigate this case. It will lead to Born's rule — to the relation between the coefficients α_k of the corresponding set of the candidate pointer states $\{|\sigma_k\rangle\}$ and their probabilities. The entangled state vector,

$$|\bar{\psi}_{S\mathcal{E}}\rangle = \sum_{k=1}^N |\alpha| e^{i\varphi_k}|\sigma_k\rangle|\varepsilon_k\rangle, \tag{6}$$

with all the coefficients of equal magnitude has a much larger set of envariant properties than Eq. (1): Now any orthonormal basis can be regarded as Schmidt. In particular, a unitary transformation diagonal in a Hadamard transform of any pair of basis states of $\{|\sigma_k\rangle\}$ generates a different looking transformation of S that consists of a (weighted) sum of an identity and a "swap":

$$u_S(i \leftrightarrow j) = e^{i\varphi_{i,j}}|\sigma_i\rangle\langle\sigma_j| + \text{H.c.}, \tag{7a}$$

Swap is envariant—it can be undone by a "counterswap":

$$u_\mathcal{E}(i \leftrightarrow j) = e^{i(\varphi_{i,j} + \varphi_i - \varphi_j + 2\pi l_{ij})}|\varepsilon_i\rangle\langle\varepsilon_j| + \text{H.c.}, \tag{7b}$$

where l_{ij} is an integer. An $i \leftrightarrow j$ swap switches $|\sigma_i\rangle$ and $|\sigma_j\rangle$. After the associated counterswap, also the states of $\mathcal{E}$ and their phases "get swapped." Thus, iff $|\alpha_i| = |\alpha_j|$,

$$[u_S(i \leftrightarrow j) \otimes u_\mathcal{E}(i \leftrightarrow j)]|\bar{\psi}_{S\mathcal{E}}\rangle = |\bar{\psi}_{S\mathcal{E}}\rangle, \tag{8}$$

which proves envariance under swaps for $|\bar{\psi}_{S\mathcal{E}}\rangle$ of Eq. (6) (and, more generally, envariance of swaps of basis states that have the same absolute values of Schmidt coefficients).

To connect envariance under swaps with probabilities, we remark that all of the states of the system described by Eq. (6) can be exchanged this way leaving the overall state unchanged. This can make no observable difference to the state of the system S alone when it is perfectly entangled [Eq. (6)] with some "environment": The joint state $|\bar{\psi}_{S\mathcal{E}}\rangle$ is envariant under swaps. When all of the coefficients of swapped states are equal, the observer with access to S alone cannot detect the effect of the swap.

Let us now make a rather general (and a bit pedantic) assumption about the measuring process: When the states are swapped, the corresponding probabilities get relabeled ($i \leftrightarrow j$). This leads us to conclude that the probabilities for any two envariantly swappable $|\sigma_k\rangle$ are equal. Moreover, when all of the orthonormal states with nonzero coefficients are swappable, and there are a total N of them, probability $p_k = p(\sigma_k)$ of each must be

$$p_k = 1/N, \tag{9a}$$

by normalization (we assume that states that do not appear in Schmidt decomposition have zero probability). Furthermore, probability of any subset with n mutually exclusive (orthonormal) $\{|\sigma_k\rangle\}$ is

$$p_{k_1 \vee k_2 \vee \cdots \vee k_n} = n/N. \tag{9b}$$

This case with equal absolute values of the coefficients was straightforward. Consider now a general case. To avoid cumbersome notation that may obscure key ideas, we focus on the case with only two nonzero coefficients:

$$|\psi_{S\mathcal{E}}\rangle = \alpha|0\rangle|\varepsilon_0\rangle + \beta|1\rangle|\varepsilon_1\rangle, \tag{10a}$$

and assume that they can be written as

$$\alpha = e^{i\varphi_0}\sqrt{m/M}; \qquad \beta = e^{i\varphi_1}\sqrt{(M-m)/M}. \tag{10b}$$

When there are no m and M for which Eq. (10b) holds exactly, we can still put upper and lower bounds on $|\alpha|$ and $|\beta|$ by taking a sequence of increasing M and $m_+ = m_- + 1$ such that $\sqrt{m_+/M} > |\alpha| > \sqrt{m_-/M}$ and, by continuity, recover our conclusions in the $M \to \infty$ limit.

The strategy now is to convert the entangled state of Eqs. (10a) and (10b) with unequal coefficients into an entangled state with equal coefficients, and then to apply envariance-based reasoning that has led to Eqs. (9a) and (9b). "Fine-graining" is a well-known trick, used on similar occasions in the classical probability, but applicable also in the quantum context [16,17]. To implement it, we need *two* systems we shall designate by C and $\mathcal{E}$ (rather than just a single environment) correlated with the "system of interest" S. There are a number of ways to motivate this split of the original $\mathcal{E}$ into "the counterweight C" and "the new $\mathcal{E}$": One can think of C (that will justify envariant swapping) and of $\mathcal{E}$ (a "second order environment" that allows one to disregard phases for reasons discussed previously) as two parts of a single "original environment" of Eq. (10a). One can also adopt a view closer to the spirit of quantum measurements and regard C as "a counter," playing a role of an apparatus. In any case, we assume that, to begin with, C with the right attributes (to be listed below) interacts with (premeasures) $\mathcal{E}$ so that the joint state has a form:

$$|\psi_{SC\mathcal{E}}\rangle \sim (e^{i\varphi_0}\sqrt{m}|0\rangle|C_0\rangle + e^{i\varphi_1}\sqrt{M-m}|1\rangle|C_1\rangle)|e_0\rangle. \tag{11}$$

Thus, states of the environment of Eq. (10a) can be rewritten as products of a state of C (already entangled with S) and of the more distant $\mathcal{E}$, (i.e., $|C_0\rangle|e_0\rangle = |\varepsilon_0\rangle$, $|C_1\rangle|e_0\rangle = |\varepsilon_1\rangle$). We now assume that $|C_0\rangle$ and $|C_1\rangle$ can be expressed in a different orthonormal basis $\{|c_j\rangle\}$:

$$|C_0\rangle = \sum_{k=1}^{m}|c_k\rangle/\sqrt{m}; \qquad |C_1\rangle = \sum_{k=m+1}^{M}|c_k\rangle/\sqrt{M-m}. \tag{12}$$

This requires the relevant subspaces of $\mathcal{H}_C$ correlated with S to have dimensions of at least m and $M-m$.

Envariance we now exploit is associated with the existence of counterswaps of $\mathcal{E}$ that undo swaps of the joint state of the composite system SC. To exhibit it, we let C interact with $\mathcal{E}$ (e.g., by employing C as a control to carry out a C-SHIFT [3]) so that $|c_k\rangle|e_0\rangle \to |c_k\rangle|e_k\rangle$, where $|e_0\rangle$ is the initial state of $\mathcal{E}$ and $\langle e_k|e_l\rangle = \delta_{kl}$. Thus,

$$|\Psi_{SC\mathcal{E}}\rangle \sim e^{i\varphi_0}\sqrt{m}\,|0\rangle\sum_{k=1}^{m}\frac{|c_k\rangle|e_k\rangle}{\sqrt{m}} + e^{i\varphi_1}\sqrt{M-m}\,|1\rangle\sum_{k=m+1}^{M}\frac{|c_k\rangle|e_k\rangle}{\sqrt{M-m}} \tag{13a}$$

obtains. $|\Psi_{SC\mathcal{E}}\rangle$ is envariant under swaps of the states $|s, c_k\rangle$ of the composite SC system (where s stands for 0 or 1, as needed) that are present (i.e., appear with a nonzero amplitude) in the $\Psi_{SC\mathcal{E}}$ above. This is made even more apparent by carrying out the obvious cancellations:

$$|\Psi_{SC\mathcal{E}}\rangle \sim e^{i\varphi_0}\sum_{k=1}^{m}|0, c_k\rangle|e_k\rangle + e^{i\varphi_1}\sum_{k=m+1}^{M}|1, c_k\rangle|e_k\rangle. \tag{13b}$$

We conclude [having repeated checks patterned on the obvious modifications of Eqs. (7a), (7b), and (8)] that $p_{0,k} = p_{1,k} = 1/M$, and that, by virtue of Eq. (9b), probabilities of $|0\rangle$ and $|1\rangle$ are

$$p_0 = \frac{m}{M} = |\alpha|^2; \qquad p_1 = \frac{M-m}{M} = |\beta|^2. \tag{14}$$

This is Born's rule. We have derived it from the most quantum of foundations—the incompatibility of the knowledge about the whole and about the parts, mandated by entanglement and embodied in envariance. Envariance shows how Born's rule arises in this purely quantum setting, i.e., without appeals to "collapse," "measurement," or any other such *deus ex machina* imposition of symptoms of classicality. Generalization to more than two states of S is straighforward.

To further clarify the implications of our derivation, we now prove that envariance also yields the *relative frequencies* interpretation of probabilities. Consider an ensemble of $\mathcal{N}$ distinguishable $SC\mathcal{E}$ triplets, all in the state given by Eq. (11). The state of the ensemble is then

$$|\Upsilon^{\mathcal{N}}_{SC\mathcal{E}}\rangle = \otimes_{\ell=1}^{\mathcal{N}}|\psi^{(\ell)}_{SC\mathcal{E}}\rangle. \tag{15}$$

We now repeat steps, Eqs. (12)–(14), for each triplet, and think of C as a counter, a detector in which states $|c_1\rangle\cdots|c_m\rangle$ of Eq. (13b) record "0" in S, while $|c_{m+1}\rangle\cdots|c_M\rangle$ record "1". Carrying out tensor product and counting terms with n detections of "0" yields the total:

$$\nu_{\mathcal{N}}(n) = \binom{\mathcal{N}}{n} m^n (M-m)^{\mathcal{N}-n}. \tag{16}$$

This immediately leads to the probability of n zeros which, for large $\mathcal{N}$, can be approximated by a Gaussian with $\langle n\rangle = |\alpha|^2\mathcal{N}$, establishing the desired link between relative frequency of outcomes and Born's rule:

$$p_{\mathcal{N}}(n) = \binom{\mathcal{N}}{n}|\alpha|^{2n}|\beta|^{2(\mathcal{N}-n)} \simeq \frac{e^{-(1/2)[(n-|\alpha|^2\mathcal{N})/(\sqrt{\mathcal{N}}|\alpha\beta|)]^2}}{\sqrt{2\pi\mathcal{N}}|\alpha\beta|}. \tag{17}$$

This strategy avoids circular use of a scalar product that invalidates [10,11] previous derivations based on Everett's "many worlds" framework [18–20].

Note that steps (11)–(14) involving the counter(weight) C do not need to be implemented — our conclusions are based on the fact that they *can* be implemented. Our derivation of probabilities is based on the nature of quantum states of joint systems (i.e., entanglement), and on the resulting envariance, rather than on details such as dimension of $\mathcal{H}_C$ or even $\mathcal{H}_\mathcal{E}$, providing that the obvious condition [Dim $\mathcal{H}_\mathcal{E} \geq$ Dim $\mathcal{H}_S$ that allows for entanglement in, e.g., $\psi_{S\mathcal{E}}$ of Eq. (1)] is met.

The setting (involving entanglement between S and $\mathcal{E}$) that has led to Born's rule is that of einselection and decoherence. Of course, as we have attempted to validate foundations of decoherence, we have not relied on it. But the very fact that Born's rule naturally obtains with the help of environment—as do many other symptoms of classicality [3–7]—adds credence to this view of the emergence of the classical. This last remark requires elaboration: One might have hoped to arrive at the probability interpretation without appealing to the environment. Indeed, there were many attempts in this vein [17–22]. I do not see how approaches that do not obliterate phases in some manner (as Refs. [2,21,22] do) can succeed in obtaining probabilities of $|\sigma_k\rangle$ in a pure state, e.g., $|\chi_S\rangle = \sum_{k=1}^N \alpha_k |\sigma_k\rangle$. Equal probabilities for $|\sigma_k\rangle$ must imply that swapping of the alternatives in the state of the form $|\bar{\chi}_S\rangle = \sum_{k=1}^N |\alpha| e^{i\varphi_k} |\sigma_k\rangle$ should not be detectable if the obvious consequence (that "all the potential outcomes are equivalent") is to follow. But this is demonstrably *not* the case. For instance, states $|\chi\rangle \sim |1\rangle + |2\rangle - |3\rangle$ and $|\chi'\rangle \sim |3\rangle + |2\rangle - |1\rangle$ are distinguishable through obvious interference measurements. Thus, any swapping would change relative phases, which in an isolated system are perfectly detectable. In brief, an observer given a suitably large ensemble of identically prepared systems will be eventually able to tell that they are in a pure state $|\chi\rangle$ (or $|\chi'\rangle$). By contrast, an observer presented with an ensemble of entangled pairs $S\mathcal{E}$ would be similarly forced to conclude that he is partially ignorant of the state of S, and that its state is given by ρ_S of Eq. (2). This ignorance arises *not* because of the inaccessibility of $\mathcal{E}$, but as a consequence of entanglement and envariance.

Envariance of entangled quantum states follows from the nonlocality of joint states and from the locality of systems, or, put a bit differently, from the coexistence of perfect knowledge of the whole and complete ignorance of the parts. This very quantum foundation provides the basis for the derivation of Born's rule. We note that, while quantum theory and the presently available data (e.g., obtained in a course of tests of Bell's inequalities) appear to be consistent with envariance, its validity has not been deliberately verified. Experiments on entangled systems that demonstrate "undoing" of an envariant transformation applied on one end of an entangled pair with a countertransformation, Eqs. (7a) and (7b), acting on the other end would be fundamentally important.

It may be surprising that the few-line derivation of Born's rule presented above has not been discovered before. I believe this is because the quantum properties underlying the proof have been usually regarded as something that needs to be explained (rather than used as a basis for an explanation). In any case, when envariance is accepted as a basic "quantum fact of life," effective classicality can be understood in a more satisfactory and fundamental way, and much of the "measurement problem" [23] can be resolved in a quantum setting.

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