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Title: The Graph Laplacian and the Dynamics of Complex Networks

Author(s): Thulasidasan, Sunil

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The Graph Laplacian and the Dynamics of Complex Networks

Sunil Thulasidasan

Computational & Statistical Sciences Division

Los Alamos National Laboratory

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Abstract

- In this talk, we explore the structure of networks from a spectral graph-theoretic perspective by analyzing the properties of the Laplacian matrix associated with the graph induced by a network. We will see how the eigenvalues of the graph Laplacian relate to the underlying network structure and dynamics and provides insight into a phenomenon frequently observed in real world networks -- the emergence of collective behavior from purely local interactions seen in the coordinated motion of animals and phase transitions in biological networks, to name a few.

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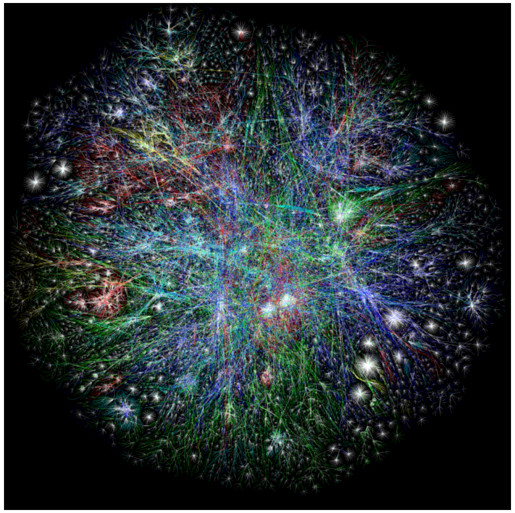
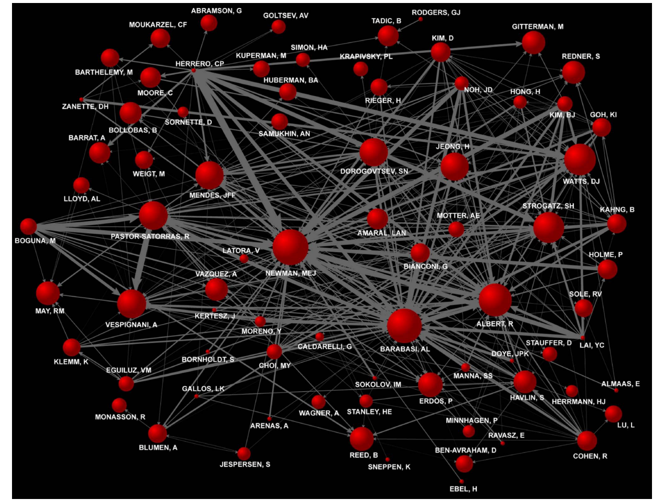
Networks



Internet
map opte.org

Citation
network



Fish
School
uwphotographyguide.com

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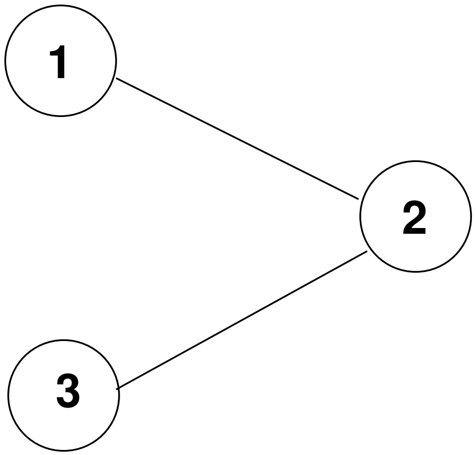
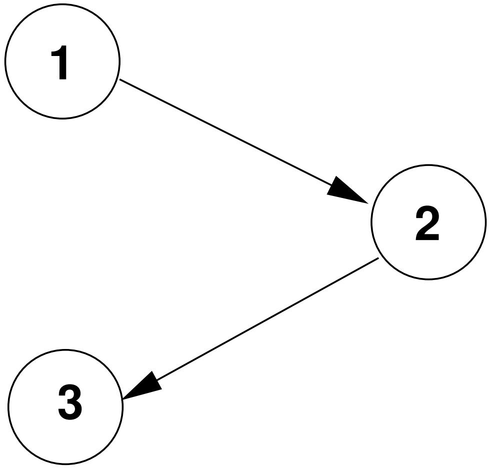
Graphs



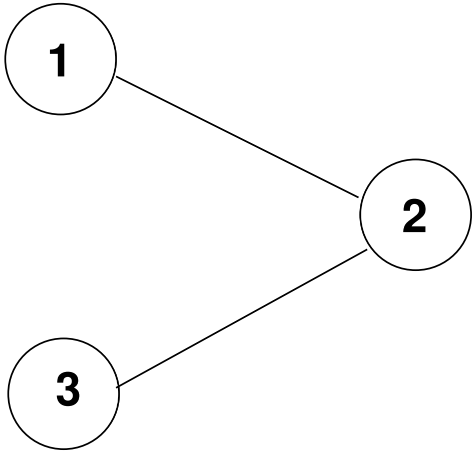


G = (V,E)
V = {1,2,3}
E = { {1,2}, {2,1},{2,3},{3,2}}

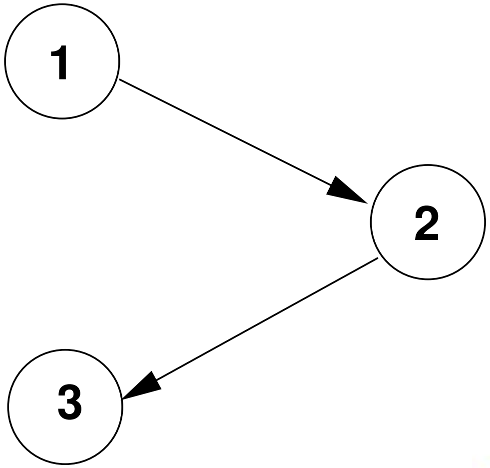
G = (V,E)
V = {1,2,3}
E = {{1,2},{2,3}}

Adjacency Matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$



$$A_D = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



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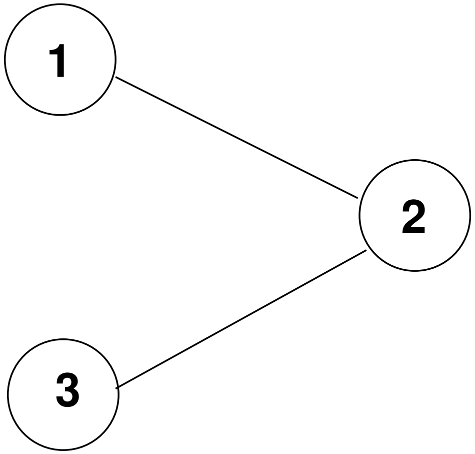
Degree matrix



$$\Delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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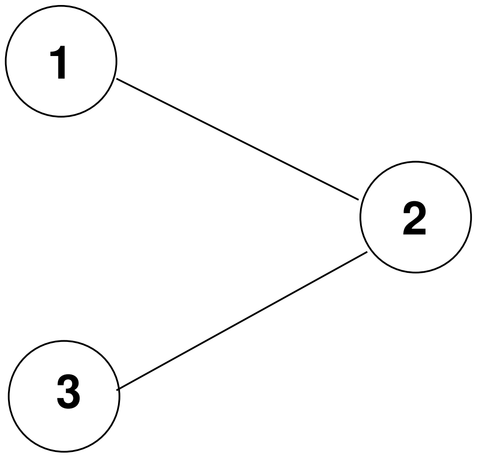
Graph Laplacian

$$\mathbf{L}(\mathcal{G}) = \mathbf{\Delta}(\mathcal{G}) - \mathbf{A}(\mathcal{G})$$

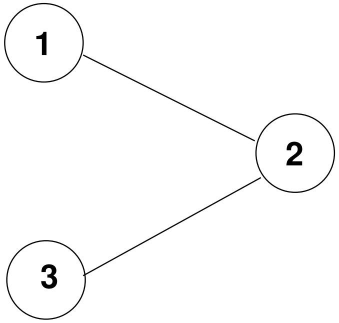
$$\Delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$



$$L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Graph Laplacian….

- Consider the following neighbor-influenced process on a network



$$\dot{x}_i = C \sum_{j \in \mathcal{N}(i)} (x_j - x_i)$$

$$\Rightarrow \dot{\mathbf{x}} = C(\mathbf{A} - \mathbf{\Delta})\mathbf{x}$$

$$\Rightarrow \dot{\mathbf{x}} + C(\mathbf{D} - \mathbf{A})\mathbf{x} = 0$$

$$\Rightarrow \dot{\mathbf{x}} + C\mathbf{L}\mathbf{x} = 0$$

$$L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Graph Laplacian….

- A consensus algorithm on networks:

$$\dot{x}_i = \sum_{j \in \mathcal{N}(i)} (x_j - x_i)$$

In compact notation, this becomes:

$$\dot{\mathbf{x}} = -L(G)\mathbf{x}$$

What can we say about the convergence of this process?

Graph Laplacian eigenvalues

$$\mathbf{L}(\mathcal{G}) = \mathbf{\Delta}(\mathcal{G}) - \mathbf{A}(\mathcal{G})$$

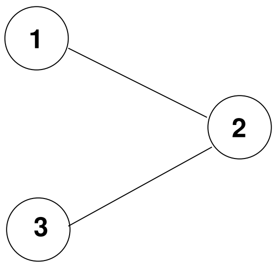
$$L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

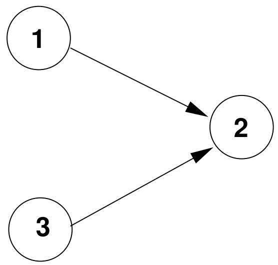
$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

0 is always an eigenvalue of the Laplacian, and the column of 1's is the Associated eigenvector

The Incidence Matrix

$$D = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{bmatrix}; D^T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$





$$DD^T = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = L$$

$$\left[D(\mathcal{G}^o) D(\mathcal{G}^o)^T \right]_{ij} = \sum_{k=1}^{m} [D(\mathcal{G}^o)]_{ij} \, [D(\mathcal{G}^o)]_{kj}$$

If $i = j$, then the product is just the degree of v_i. Else -1 if an edge exists between v_i and v_j and 0 otherwise. Thus we get back the Laplacian.

Laplacian properties

$$\begin{aligned} \mathbf{L}(\mathcal{G}) &= \mathbf{D}(\mathcal{G})\mathbf{D}(\mathcal{G})^{\mathbf{T}} \\ \Rightarrow \mathbf{x}^{\mathbf{T}}\mathbf{L}(\mathcal{G})\mathbf{x} &= \mathbf{x}^{\mathbf{T}}\mathbf{D}(\mathcal{G})\mathbf{D}(\mathcal{G})^{\mathbf{T}}\mathbf{x} \\ &= (\mathbf{D}(\mathcal{G})^{\mathbf{T}}\mathbf{x})^{\mathbf{T}}(\mathbf{D}(\mathcal{G})^{\mathbf{T}}\mathbf{x}) \\ &= ||\mathbf{D}(\mathcal{G})^{\mathbf{T}}\mathbf{x}||_{\mathbf{2}}^{\mathbf{2}} \end{aligned}$$

Then, the Laplacian is positive semi-definite with at least one zero eigenvalue

$$0 = \lambda_1(G) \leq \lambda_2(\mathcal{G}) \leq \ldots \lambda_n(G)$$

Back to consensus…

$$\dot{\mathbf{x}} = -L(G)\mathbf{x}$$

$$\mathbf{x(t)} = \mathbf{e}^{-\mathbf{L(G)t}}\mathbf{x_0}$$

$$\mathbf{\Lambda}(\mathcal{G}) = \mathbf{Diag}([\lambda_1(\mathcal{G}), \ldots, \lambda_n(\mathcal{G})]^\mathbf{T})$$

$$\mathbf{x(t)} = \mathbf{e}^{-\mathbf{L(G)t}}\mathbf{x_0}$$

$$\begin{aligned}\mathbf{e}^{-\mathbf{L(G)t}} &= \mathbf{e}^{(\mathbf{U\Lambda(G)U^T})\mathbf{t}} \\ &= \mathbf{U}\mathbf{e}^{-\mathbf{\Lambda(G)}}\mathbf{U^T}\end{aligned}$$

$$= \mathbf{e}^{-\lambda_1 \mathbf{t}}\mathbf{u_1 u_1^T} + \mathbf{e}^{-\lambda_2 \mathbf{t}}\mathbf{u_2 u_2^T} \ldots \mathbf{e}^{-\lambda_n \mathbf{t}}\mathbf{u_n u_n^T}$$

When would $\lambda_2 = 0$?

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λ_2 and graph connectedness

$$A = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & & A_c \end{bmatrix}$$

$$L = \begin{bmatrix} L_1 & 0 & \dots & 0 \\ 0 & L_2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & & L_c \end{bmatrix}$$

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λ_2 and graph connectedness

$$\begin{bmatrix} L_1 & 0 & \dots & 0 \\ 0 & L_2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & & L_c \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_c \end{bmatrix} = 0$$

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_c \end{bmatrix} = \alpha_1 \begin{bmatrix} \mathbb{1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ \mathbb{1} \\ \vdots \\ 0 \end{bmatrix} + \dots \alpha_c \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \mathbb{1} \end{bmatrix}$$

Thus, number of connected components is dimension of null space of L

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Flocking/swarming, coupled oscillators etc

Image from National Geographic

$$u_i = \frac{k}{n}\sum_{j=1}^{n}(\theta_j - \theta_i)$$

$$u_i = \frac{k}{n}\sum_{j=1}^{n}\sin(\theta_j - \theta_i)$$

- For small phase differences, this linearizes to consensus model.
- For non-linear systems (coupled oscillators) can use positive-semi-definite property of Laplacian to construct potential functions.

Discrete case consensus

Does this converge?

$$\mathbf{x}_{(\mathbf{k+1})} = \mathbf{x}_{\mathbf{k}} + \Delta t \sum_{j \in N(i)} (\mathbf{x}_{\mathbf{j}}(\mathbf{k}) - \mathbf{x}_{\mathbf{i}}(\mathbf{k}))$$

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Discrete case consensus

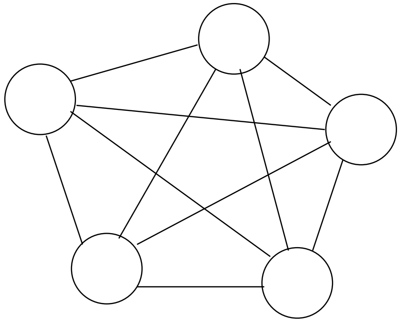
$$\mathbf{x}_{(\mathbf{k}+1)} = \mathbf{x}_{\mathbf{k}} + \Delta t \sum_{j \in N(i)} (\mathbf{x}_{\mathbf{j}}(\mathbf{k}) - \mathbf{x}_{\mathbf{i}}(\mathbf{k}))$$
$$= (I - \Delta t L(\mathcal{G}))\mathbf{x}_{\mathbf{k}}$$

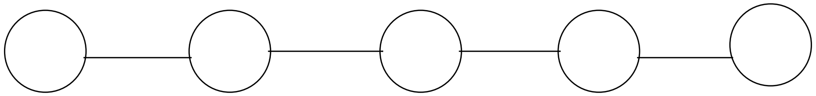
$$\text{Let } I - \Delta t L(\mathcal{G}))\mathbf{x}_{\mathbf{k}} = M$$

We need

$$-1 < 1 - \Delta t \lambda_i(G) \leq 1$$
$$\Rightarrow 0 < \Delta t \lambda_{max}(G) < 2$$
$$\Rightarrow 0 \leq \Delta t < \frac{2}{\lambda_{max}(G)}$$

λ_2 and graph structure



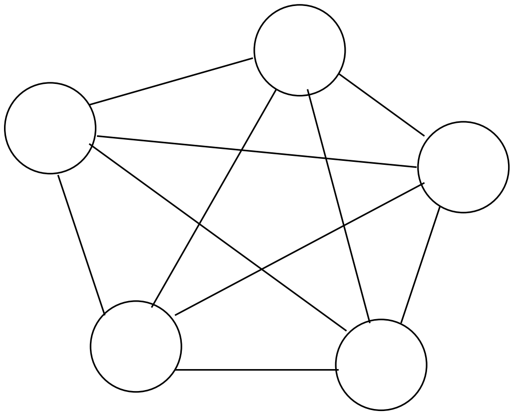


$\lambda_2 = 0.38$

$\lambda_2 = 5$

λ_2 is an indication of graph diameter

λ_2 and graph resiliency



$$\lambda_2(G) \leq \kappa_0(G) \leq \kappa_1(G) \leq d_{min}(G)$$

where

- $\kappa_0(G)$ is the node connectivity
- $\kappa_1(G)$ is the edge connectivity
- d_{min} is the minimum degree in G.

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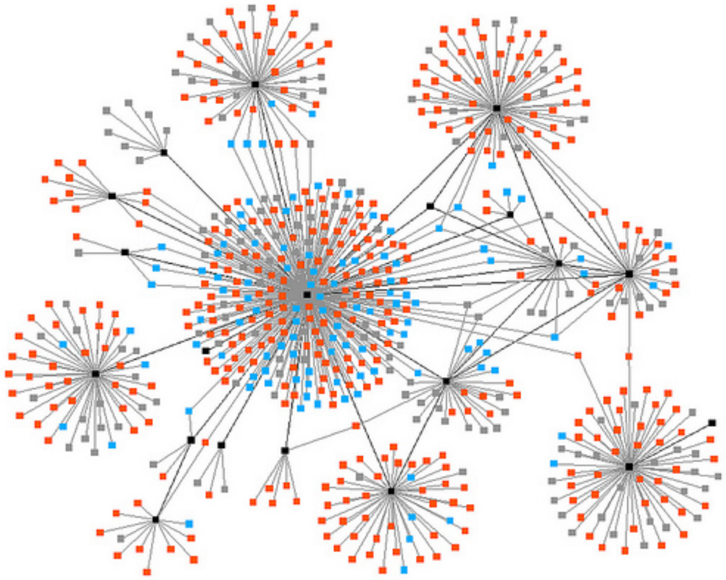
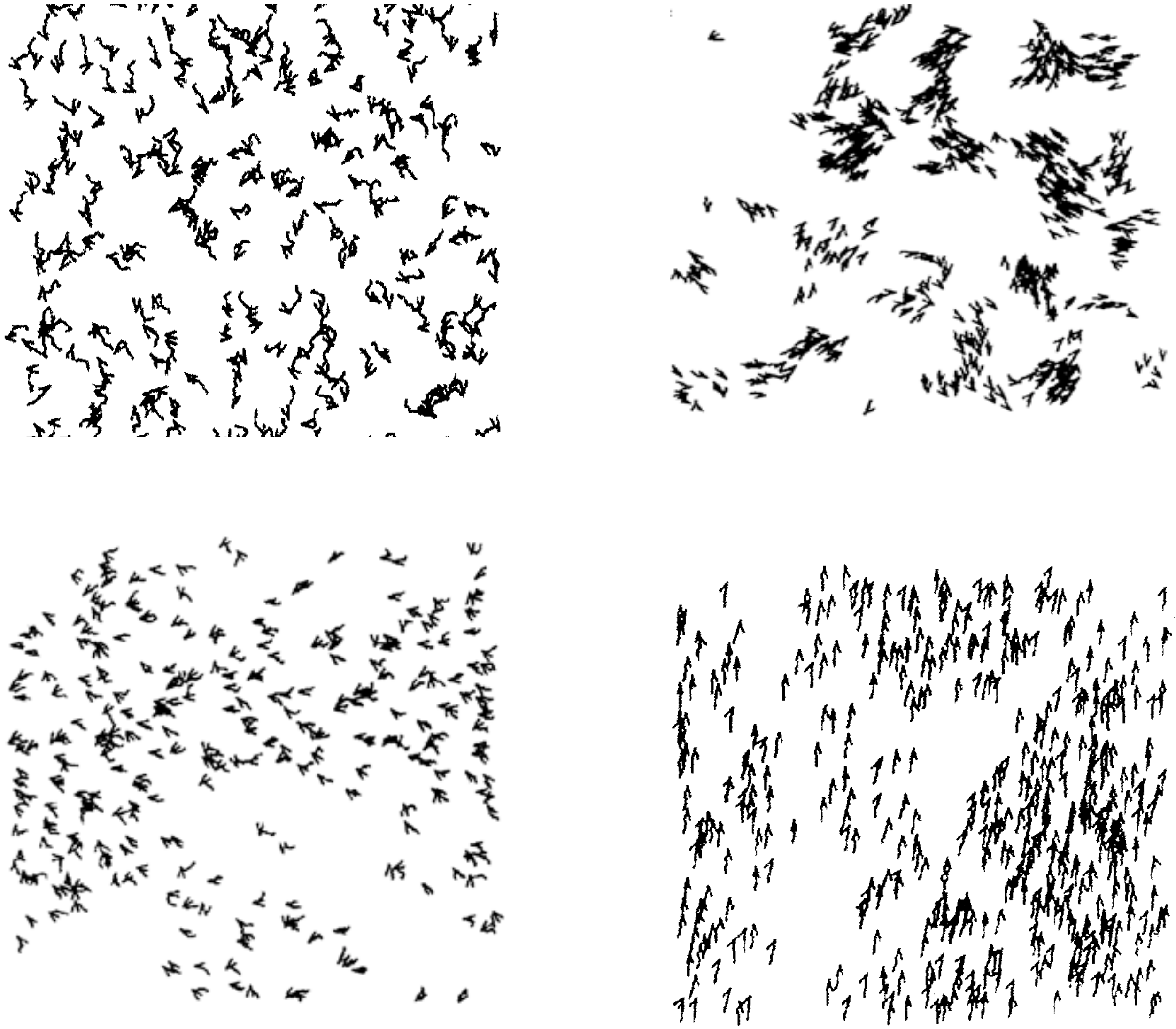
Small World Networks



- Characterized by low diameter
- High edge resilience
- Fast convergence rate
- In other words, high λ_2

Vicsek et al.: “ **A Novel Type of Phase Transition in a System of Self-Driven Particles**” *Phys. Rev. , Aug 1995*



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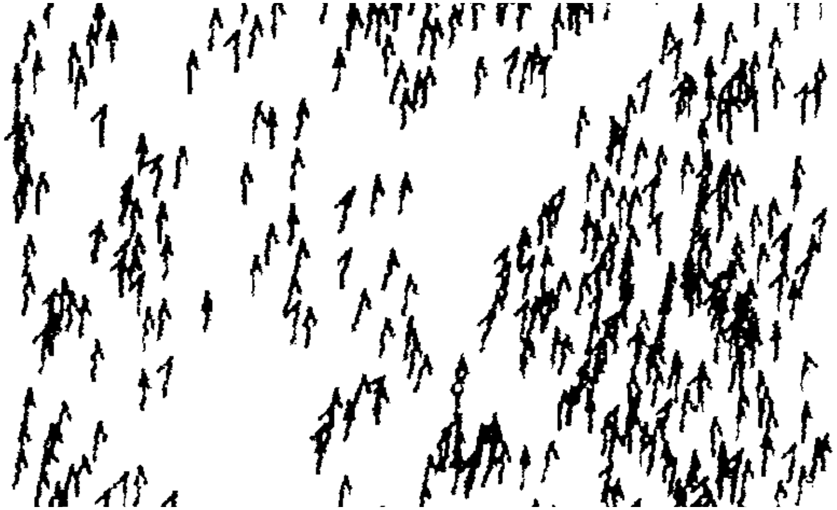
Phase transition in physical systems



"… only an extremely unusual crossover could change this tendency. A plausible physical picture.. .. is that there is effective (long range) interaction radius" (*Vicsek 1995*]

Combinatorial phase transition (emergence of connectedness) leads to a physical phase transition.

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