Statistics of intermediate duration averages of atmospheric scintillation

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ABSTRACT

An extensive set of measurements was recently obtained, of the scintillation of a laser propagated over long horizontal paths through atmospheric turbulence, at altitudes spanning the tropopause. These measurements were made over sequences of parallel but displaced paths, like the rungs of a ladder. It is shown here that the intensity reductions due to scintillation of two parallel paths separated by 35.6 meters are partially correlated. Further, the correlations between paths with the discrete experimental separations are used to construct the correlation functions for arbitrary path displacement. The variance in continuous moving averages of the relative intensity is then found in terms of the correlation functions, parameterized by the distance the propagation path is swept through the turbulence. An empirical formulation is developed for use in assessing the expected distribution of intensity reductions in various laser systems. This analysis recovers the statistics of atmospheric scintillation for the important regime in between the two extremes of a snapshot and a long time average.

1. INTRODUCTION

Light propagating through atmospheric turbulence undergoes several phenomena that degrade the attainable intensity or resolution from their diffraction limits. One of these, scintillation, generates a perturbation on the amplitude of the electro-magnetic field, that depends on position in the transmitting or receiving aperture. When the field amplitude contributions from all parts of the aperture are added, the resulting target intensity is found to differ from the diffraction-limited value.

The scintillated intensity depends on the instantaneous realization of the index of refraction perturbations generated along the propagation path by the turbulence. As the propagation path traverses the turbulence, the scintillated intensity changes. Define S as the instantaneous intensity divided by the intensity that would have been found in the absence of scintillation. As the turbulent atmosphere moves due to winds, S follows a log-normal distribution, which is to say that the log of S follows a normal distribution.

For applications where the illumination is essentially instantaneous (as in pulsed laser applications), the distribution of delivered intensity simply follows the log-normal distribution. For applications where the illumination lasts a long time, or where many pulses are employed, the effective delivered intensity is simply the mean of the log-normal distribution of S, also known as the scintillation Strehl.

For applications where the illumination takes place over an intermediate time, however, neither of these limits apply. The scintillation Strehl averaged over a finite time will also be a statistically distributed quantity. The distribution of the finite-time average scintillation Strehl has been examined by statistical analysis of nearly 100,000 measurements of S, taken in the ABLEX program.

The variance of the finite time average Strehl distribution is characterized by the variance of the distribution of the instantaneous S, the long time average Strehl, and the distance traversed by the propagation path during the illumination. An empirical fit for these distributions is developed in this paper.

2. THE AIRBORNE LASER SCINTILLATION MEASUREMENTS (ABLEX)

Beginning in January 1993, Phillips Laboratory carried out a series of measurements of scintillation for propagation across long paths through the atmosphere[1-4]. In the experiment, a laser was emitted from one airplane (Harp), and the intensity profile was measured in a pupil plane scintillometer on a second airplane (the modified NC135 Argus). The pair made eight flights. During each flight, there were a number of measurement series. For each series, both planes were held at constant altitude, and maintained a constant separation. The altitudes ranged from 330 to 481 hundred feet above sea level, and the separation ranged from 23 to 200 km. The series ranged from 10.5 to 608 seconds in duration. During each series, the laser on Harp was pulsed at six pulses per second. Each pulse produced a frame of data, consisting of the intensity measured at each pixel in the scintillometer on Argus. In addition, for each measurement of the pulsed laser intensity profile, there was another measurement with the laser off, of the background noise level. At plane speeds of 415 knots, the propagation path of consecutive frames was displaced by 35.6 meters.

In total there were 176 data series taken, with 98,486 frames of laser intensity profile measurements. A subset of this data was constructed for this analysis, consisting of the 63 series which extended over 90 seconds or more. This subset includes 67,012 of the data frames.

The instantaneous relative intensity S for a frame was obtained by squaring the aperture average of the square root of the measured pixel intensities, and then normalizing by the square of the aperture average of the square root of the series average pixel intensity. This normalization cancels the effects of variations in the atmospheric absorption and scattering, and beacon brightness. Phase perturbations have no effect on this relative intensity either, so this scintillation Strehl is the same as would be obtained in a system employing perfect phase compensation[2].

3. STATISTICS OF S

Each series produced a sequence of S values, which is denoted by the set $\{S_i\}$. The propagation path of consecutive measurements was displaced by 35.6 meters. Fig. 1 shows the 1465 consecutive values of S_i (spanning over 52 km) for the 20th series of the 7th flight.

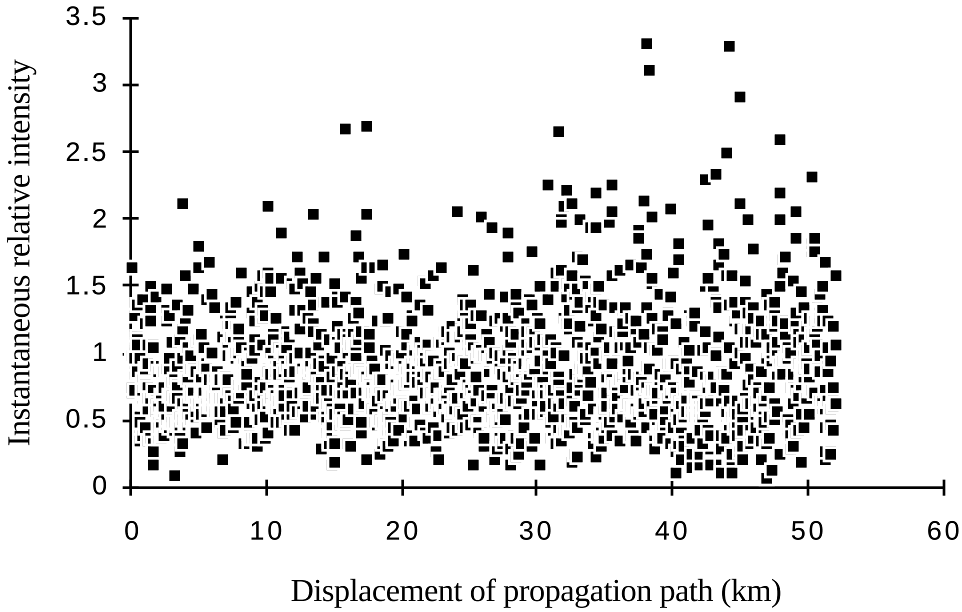

Figure 1. Measured instantaneous intensity reduction due to scintillation for the 20th series of the 7th ABLEX flight. Measurements were taken every 35.6 meters along a flight of 52.15 km. The average intensity reduction is 0.8673, and the variance is 0.1876.

For this series, the mean value of $\{S_i\}$ is 0.8673, and the variance of $\{S_i\}$ is 0.1876. The distribution of $\{S_i\}$ is found by sorting the values in the set. The transpose of a plot of these sorted values can then be interpreted as a plot of the cumulative probability distribution. The distribution of $\{S_i\}$ is shown in Fig. 2, again using the 20th series of the 7th flight. Also shown is the cumulative probability for a log normal distribution with the same mean and variance. As was found for all 63 series, the distribution of $\{S_i\}$ is roughly log normal.

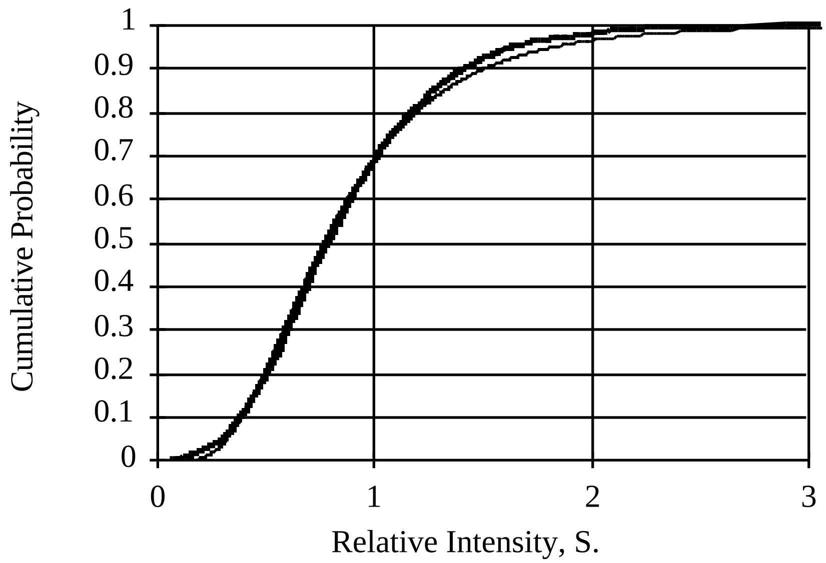

Figure 2. Cumulative probability distribution of the relative intensity data {Si}, from the 1465 measurements in the 20th series of the seventh flight. The log-normal distribution with variance of 0.1876 and mean of 0.8673 is also shown.

In Fig. 3, the variance of $\{S_i\}$ is shown as a function of the series Strehl, for the 63 series which extended 90 seconds or more. The series Strehl, $\bar{S}$, is the mean of $\{S_i\}$. The $\{S_i\}$ values are sometimes more and sometimes less clustered about the mean, depending on the particular series examined.

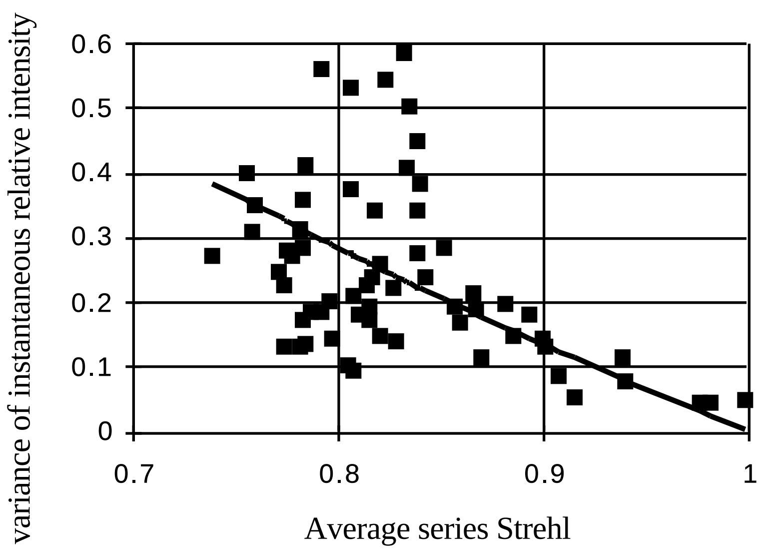

Figure 3. The variance of the instantaneous relative intensity for the 63 ABLEX series with at least 90 seconds duration, as a function of the series Strehl. The fit of Eq[1] is shown as the solid line.

The variance of $\{S_i\}$ can be represented by the fit (least squares weighted by the number of frames in each series)

$$\mathrm{var}\{S_i\}_{FIT} = 1.28\ln(1/\bar{S}) \qquad (1)$$

This fit is also shown in Fig. 3, as the solid line.

4. STATISTICS OF MOVING AVERAGE OF S

A graph of the moving average of consecutive frames of ABLEX data illuminates the physics of the problem. In Fig. 4, the moving average over 120 frames (4272 meters) is shown, for the 20th series of the 7th flight. It is clear that the relative intensity is not a stationary random function: there are slow variations in the mean relative intensity that have distance scales up to 10 kilometers.

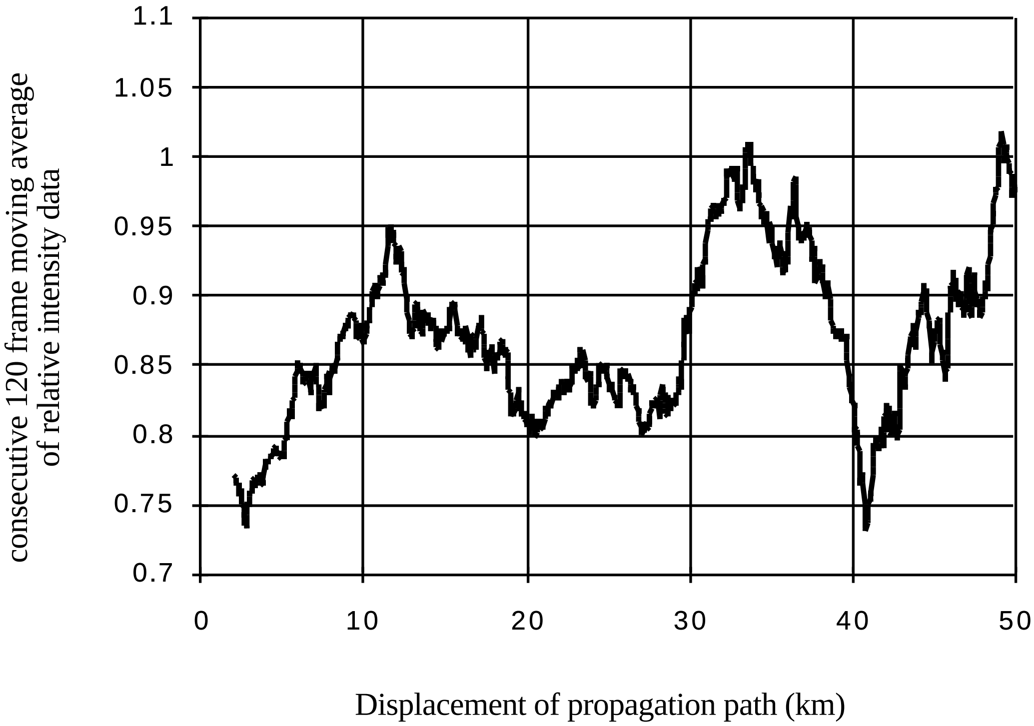

Figure 4. The moving 120 frame consecutive average of the relative intensity, for the 20th series of the 7th ABLEX flight.

The objective here is to determine the statistics of continuous averages of S over finite times or displacements. Let S(x) be the intensity reduction due to scintillation. x is a coordinate which gives the displacement of the propagation path. For the airborne applications, x corresponds to motion of the plane normal to the propagation path. For ground to space applications, x corresponds to the distance that the turbulent atmosphere moves due to wind. The average of S(x) for a continuous illumination in which the propagation path moves from x_o to x_o+X, denoted $A(X;x_o)$, is given by

$$A(X;x_o) = \frac{1}{X}\int_{x_o}^{x_o+X} dx S(x) \qquad (2)$$

It is the distribution of $A(X;x_o)$ that we want to find. Plots of the distribution of the moving average of $\{S_i\}$ from the ABLEX data show that the distribution of $A(X;x_o)$ can be taken to be log normal. The distribution of $A(X;x_o)$ is completely specified by var(A(X)), the mean of S(x), and the assertion that it is log normal.

The variance of $A(X;x_o)$ can be found in terms of the correlation function of S(x). The correlation function[5] of S(x) is defined as

$$B_S(\xi) = \left\langle (S(x_o) - \bar{S})(S(x_o + \xi) - \bar{S}) \right\rangle \tag{3}$$

where $\bar{S} = \langle S(x) \rangle = \lim_{X \to \infty} \frac{1}{X} \int_0^X dx S(x)$ is the expected value of S(x). The correlation function can be evaluated from each ABLEX series, using Eq[3]. Figure 5 shows the normalized correlation function of the relative intensity obtained from the 20th series of the 7th flight, where the normalized correlation function is

$$r(\xi) = \frac{B_S(\xi)}{B_S(0)} \tag{4}$$

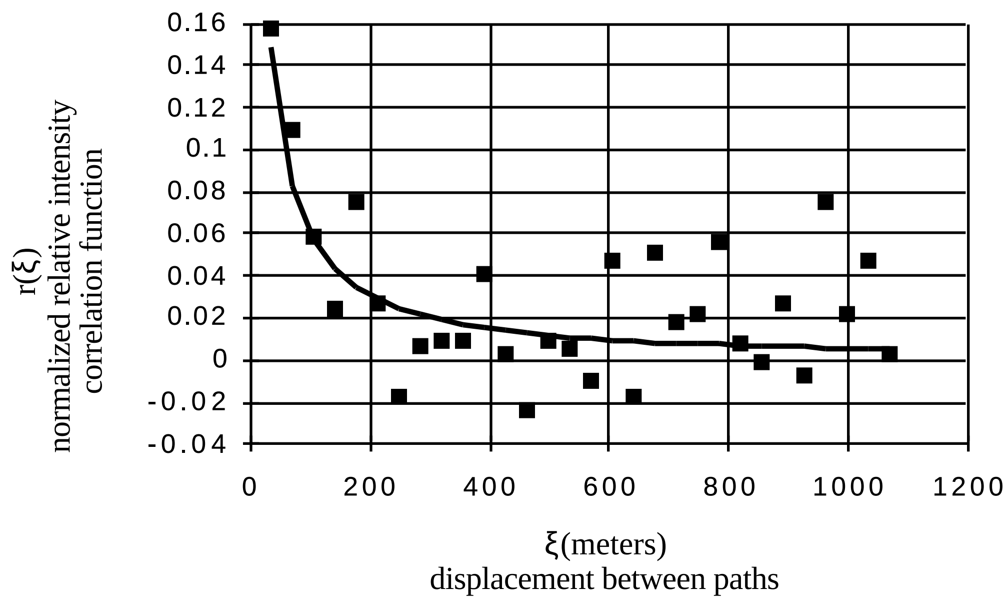

Figure 5. The normalized relative intensity correlation function, for the 20th series of the 7th ABLEX flight. The fit of Eq[5] is shown as the solid line.

For this series, the normalized covariance can be fit (least squares) as a function of ξ in meters, by

$$r_{i20}(\xi) \quad \left(1 + (\xi / 16.88)^{0.63}\right)^{-2} \tag{5}$$

This fit is also shown in figure 5.

We next derive the variance of $A(X;x_o)$ in terms of the correlation function. The variance of $A(X;x_o)$ is

$$\mathrm{var}(A(X)) = \left\langle (A(X;x_o) - \bar{S})^2 \right\rangle \tag{6}$$

The angle brackets indicate the expectation value over all x_o. Now substitute Eq[2] into Eq[6] and convert the square of the integral into a double integral:

$$\mathrm{var}(A(X)) = \left\langle \frac{1}{X^2} \int_{x_o}^{x_o+X} dx' \int_{x_o}^{x_o+X} dx''(S(x') - \bar{S})(S(x'') - \bar{S}) \right\rangle \tag{7}$$

Noting the symmetry of the integrand allows the integral to be split into two halves:

$$\mathrm{var}(A(X)) = \left\langle \frac{2}{X^2} \int_{x_o}^{x_o+X} dx' \int_{x'}^{x_o+X} dx''(S(x') - \bar{S})(S(x'') - \bar{S}) \right\rangle \tag{8}$$

Next substitute $\xi = x'' - x'$. This gives

$$\mathrm{var}(A(X)) = \left\langle \frac{2}{X^2} \int_{x_o}^{x_o+X} dx' \int_{0}^{x_o+X-x'} d\xi\ (S(x') - \bar{S})(S(x'+\xi) - \bar{S}) \right\rangle \tag{9}$$

Next, exchange the order of integration, giving

$$\mathrm{var}(A(X)) = \frac{2}{X^2} \int_0^X d\xi \left\langle \int_{x_o}^{x_o+X-\xi} dx'(S(x') - \bar{S})(S(x'+\xi) - \bar{S}) \right\rangle \tag{10}$$

Since the expectation value washes out the dependence of the integrand on x', the integral over x' is trivial:

$$\mathrm{var}(A(X)) = \frac{2}{X^2} \int_0^X d\xi (X - \xi) \left\langle (S(x_o) - \bar{S})(S(x_o + \xi) - \bar{S}) \right\rangle \tag{11}$$

or in terms of the correlation function

$$\mathrm{var}(A(X)) = \frac{2}{X^2} \int_0^X d\xi (X - \xi) B_S(\xi) \tag{12}$$

For X=0, this gives var(A(X)) = var(S(x)) = $B_S(0)$. The normalized variance of the continuous moving average, given by the ratio of the variance of the moving average to the variance of the instantaneous relative intensity can be then be defined:

$$R(X) \equiv \frac{\mathrm{var}(A(X))}{\mathrm{var}(A(0))} = \frac{2}{X^2}\int_0^X d\xi\,(X-\xi)r(\xi) \tag{13}$$

This integral can be discretized into a sum by dividing X into m subintervals:

$$R(X) = \frac{2}{m^2}\sum_{i=1}^{m}(m-i)r(iX/m) + \frac{2}{X^2}\int_0^{X/2m} d\xi\,(X-\xi)r(\xi) \tag{14}$$

The summation term of Eq[14] gives the integral of Eq[13] from ξ=X/2m to ξ=X. It can be evaluated directly with the ABLEX discrete correlation function, such as that shown in Fig. 5. The integral term of Eq[14] gives the integral of Eq[13] from ξ=0 to ξ=X/2m. It can be evaluated using a fit to the correlation function for the particular ABLEX series, such as that given by Eq[5]. While the ABLEX data allows only a crude estimate of the correlation function for ξ<X/2m, the integral term of Eq[14] is found to be insensitive to the functional form of the fit to the normalized correlation function, as long as r(0)=1, the derivative of r at 0 is 0, and the fit agrees well with the first few discrete values of the observed normalized correlation function.

For the 20th series of the 7th flight, for X=640.8 meters (i.e. m=18), the summation term of Eq[12] gives 0.044. When the fit of Eq[5] is substituted into the integral term of Eq[14], the integral can be performed to give 0.022. The variance of the 640.8 meter average relative intensity is thus 0.066 times the variance of the instantaneous relative intensity, for this series, which product is 0.0124.

The summation term of Eq[14] can be obtained in terms of the variance of moving averages of ABLEX data. Let $\{Tn_i\}$ be the set of N-n moving averages over n consecutive frames of $\{S_i\}$, where $\{S_i\}$ is the set of N consecutive values of S from an ABLEX series. The ith member of the set is thus

$$Tn_i = \frac{1}{n}\sum_{j=i+1}^{n} S_i \tag{15}$$

The variance of $\{Tn_i\}$ for a given ABLEX series is then

$$\mathrm{var}\{Tn_i\} = \left\langle \left(\frac{1}{n}\sum_{j=i+1}^{i+n}(S_j - \bar{S})\right)^2 \right\rangle \tag{16}$$

where the ensemble average now denotes an average over the N-n possible values of i in the ABLEX series. By means of a discretized version of the above derivation, there follows that

$$\frac{2}{n^2}\sum_{k=1}^{n}(n-k)B_E(kX/n) = \mathrm{var}\{Tn_i\} - \mathrm{var}\{S_i\}/n \tag{17}$$

The interpretation of Eq[17] is that the first term of Eq[14] is equal to the variance of the discrete moving average minus that part of the discrete moving average variance that is due to the statistics of the finite sample.

The normalized variance of the continuous moving average has been evaluated as a function of X, for the 20th series of the 7th flight, and is shown in Fig. 6. Finally, we want to combine the results of the 63 ABLEX series which extended over 90 seconds or more. Figure 7 shows the variance of the 640.8 meter continuous moving average as a function of the mean of $\{S_i\}$, for these 63 ABLEX series. For X=640.8 meters, the average ratio over all series of var(A(X)) to var{S} is 0.0835. The line giving $0.0835*1.28\ln(1/\bar{S})$ is also shown on Fig. 7. This fit also provides the best fit (least squares) of the data.

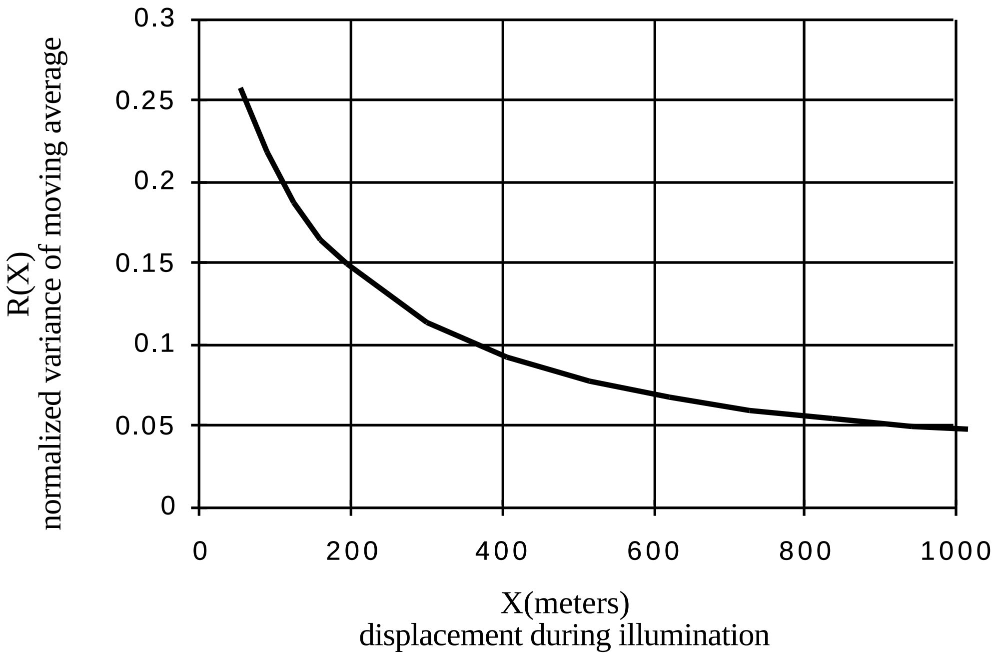

Figure 6. The normalized variance of the moving average of the relative intensity, as a function of the displacement of the propagation path during the illumination, for the 20th series of the 7th flight.

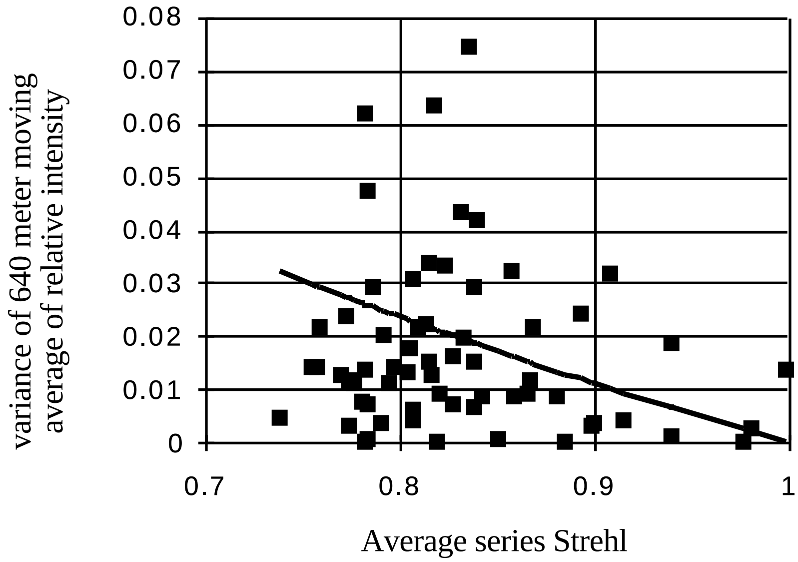

Figure 7. The variance of the continuous 640.8 meter moving average of the relative intensity, obtained from the 63 ABLEX series with duration of 90 seconds or more. Also shown is the fit described in the text.

The normalized variance of the continuous moving average has been evaluated as a function of X for all the series. Figure 8 shows the average (weighted by the number of frames) over all series of R(X) tabulated against X. Also shown is a fit to R(X), namely

$$R(X)_{FIT} = \frac{\mathrm{var}(A(X))}{\mathrm{var}\{S_i\}} = \frac{1}{\left(1+(X/58.5)^{0.377}\right)^2} \qquad (18)$$

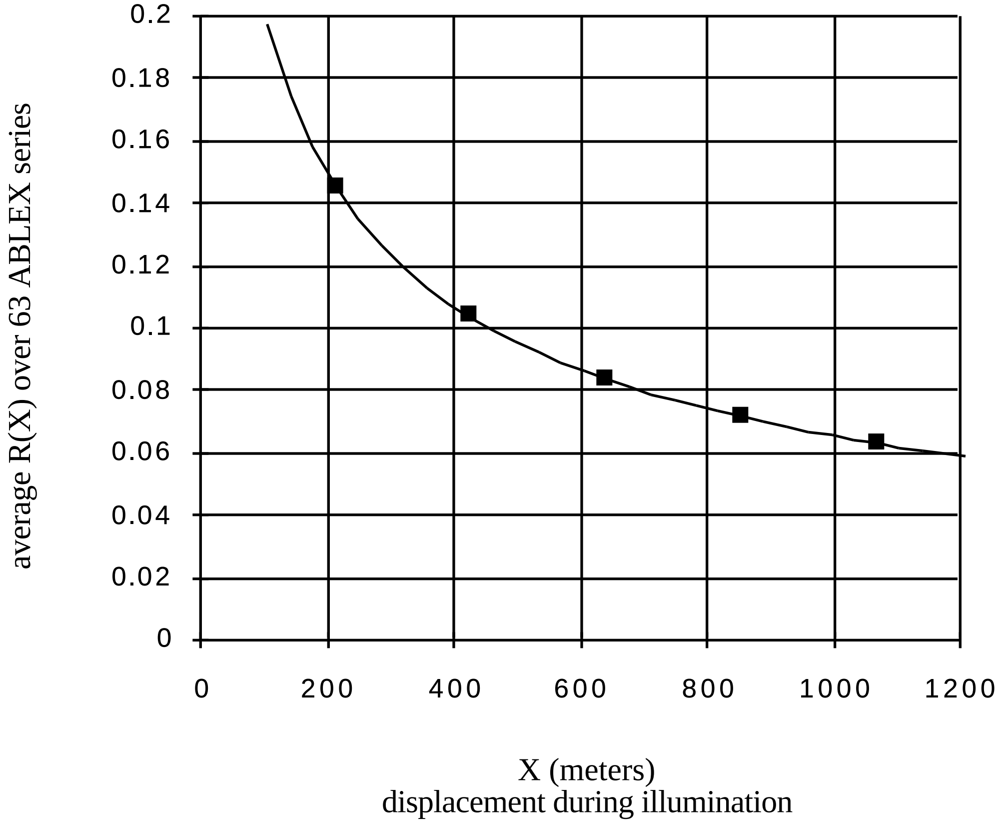

Figure 8. The normalized variance of the continuous moving average of the relative intensity, averaged over the 63 ABLEX series with duration of at least 90 seconds, as a function of the displacement in the moving average. Also shown is the fit of Eq[18].

5. APPLICATION OF FORMULATION

These calculations were performed in order to estimate the distribution of scintillation Strehls for a few second illumination across an airborne path. Previously, the approach was to calculate the expected Strehl by performing the appropriate integrals over an averaged turbulence profile[6], and use that in a deterministic fashion. As seen above, however, the time averaged Strehl is a random variable. The following algorithm could be used to implement the statistical nature of the time averaged Strehl. The expected Strehl is first evaluated, as before. We then estimate the nominal illumination time, based on the expected Strehl, the laser power, the required fluence on target, etc. From the plane speed and the nominal illumination time, we find the nominal distance that the propagation path would move through the turbulence. The representative variance of the moving average is then evaluated, using Eq[1] and Eq[18]. A random number c in (0,1) is then generated. The value of the inverse log normal distribution with the required variance, corresponding to the cumulative probability c, then gives the effective Strehl for that particular realization. Sometimes the system will perform better than average, sometimes worse.

Since A(X) has a log normal distribution, ln(A(X)) has a normal distribution. The variance of the distribution of ln(A(X)) is readily seen to be $\ln(\mathrm{var}(A(X))/\bar{S}^2 - 1)$, and the mean value of ln(A(X)) is

$\ln(\bar{S})$ minus half the variance of ln(A(X)). The inverse of this normal distribution as a function of the cumulative probability, can be used to facilitate the evaluation of the realized Strehl values.

As an example, suppose we were doing a 3 second illumination of a target from an airplane moving at 240 m/s, and we calculate that for the particular propagation path the expected scintillation Strehl is 0.75. The estimate of the variance of the instantaneous relative intensity from Eq[1] is 0.368. For a 3 second illumination at 240 m/s, the propagation path traverses a distance X of 720 meters. From Eq[18], the variance of the 720 meter average is found to be 0.0784 times 0.368, which is 0.029. We generate a random number and get 0.16. The value of a log normally distributed variable with variance .029 and cumulative probability 0.16 is 0.585. For this realization, the 3 second average Strehl is 0.585 instead of the expected value of 0.75, and the required dwell time would increase to 3.8 seconds. If we had got the random number 0.84, the realized Strehl would be 0.914. Note that these particular values of c correspond to plus and minus one standard deviation in the distribution of ln(A(X))

This methodology can be applied to other phenomena of propagation through atmospheric turbulence, especially in applications of adaptive optics compensation. For ground based applications, the wind and target motion, rather than the laser motion, is the source of the path displacements through the turbulence. The moving average of phase error Strehl ratios are also statistically distributed quantities. These include deformable mirror fitting error, anisoplanatic phase errors due to beacon-beam mismatch, errors due to finite bandwidth of the compensation system, errors due to the finite speed of light, etc.

6. REFERENCES

1. D. E. Spreen, C. B. Hogge,”Characterizing high-altitude horizontal path optical propagation,” *Proc. SPIE Laser Beam Propagation and Control* , vol. 2120 , pp 2-9 (1994).
2. R. R. Butts and L.D.Weaver, "Airborne Laser Experiment (ABLEX): Theory and Simulation," *Proc. SPIE Laser Beam Propagation and Control* , vol. 2120 , pp 10-29 (1994).
3. L. D. Weaver and R.R.Butts, "ABLEX High Altitude Laser Propagation Experiment," *Proc. SPIE Laser Beam Propagation and Control* , vol. 2120 , pp 30-42 (1994).
4. D. T. Kyrazis et. al., ”Measurement of optical turbulence in the upper troposphere and lower stratosphere,” *Proc. SPIE Laser Beam Propagation and Control* , vol. 2120 , pp 43-55 (1994).
5. V. I. Tatarskii, *Wave Propagation in a turbulent Medium*, McGraw-Hill, New York (1961).
6. P. D. Stroud, "Diffraction and Scintillation of Laser Beams by Atmospheric Turbulence," *Los Alamos National Lab Report LAUR-93-1401* (1993).