

A Variational Free-Lagrange Method for Shallow Water

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This article presents a variational free-Lagrange (VFL) method for rotating shallow water. This method was first derived by Augenbaum (2) who discretised Hamilton's action principle with a free-Lagrange data structure. The novel feature of our approach is to exploit the geometric structure preserved by the VFL method to the effect of conserving energy over long-time simulations. Numerical results demonstrate this property and motivate the extension of this approach to other variational formulations of climate models.

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1 Introduction

The variational approach The derivation and analysis of conservative numerical methods for global climate modeling is a challenging and active area of research. The traditional approach to deriving such methods requires the numerical analyst to exercise some level of their own bias. To remedy this and systematically derive the numerical method from a conferred mathematical understanding, Salmon (1) pursues the *variational approach* for geophysical modeling. The power of the variational approach rests upon Noether's theorem which states that each continuous symmetry of Hamilton's action principle exhibits a corresponding conservation law. Consequently, the numerical analyst need only choose a suitable discretisation of Hamilton's action principle, with the confidence that not only will the resulting semi-discrete equations of motion preserve geometric structure, manifesting in energy conservation, but that these equations will conserve additional quantities corresponding to the symmetries of the discrete action principle.

Voronoi diagram Following (2), we firstly outline a variational free-Lagrange (VFL) method for rotating shallow water. A distinguishing feature of the free-Lagrange method is the use of a Voronoi diagram to represent the layer thickness. The Voronoi diagram is attractive from both a practical and theoretical perspective. Firstly the approach is completely free from *mesh-tangling* problems (see 3) which compromise the performance of many of the mesh-based Lagrangian methods. Additionally, (4) shows that the corresponding discrete Hamilton's action principle permits translational and rotational symmetries and hence respective linear and angular momentum conservation laws.

Symplectic integrator Our contribution is to exploit the symplectic structure preserved by the VFL method by using a symplectic time-stepper (see 5, for a review of symplectic methods for particle methods). We demonstrate by numerical experiment that the combination of the VFL method with an explicit symplectic time-stepper exhibits no secular drift in the energy error. This property together with the computational tractability renders this approach suitable for long-time geophysical simulations.

2 The Free-Lagrange Method

Consider a particle representation of a fluid in which N^2 particles or *sites* $\mathbf{X} = \{\mathbf{X}_1, \dots, \mathbf{X}_{N^2}\}$ are individually labelled by α . Each particle is inside a polygon containing the set of points \mathbf{X}_β closer to \mathbf{X}_α than to any other site. The polygon is referred to as a *Voronoi cell* and the set of all these closed cells is the *Voronoi diagram*. A hexagonal Voronoi cell is shown in Figure 1 together with the specification of a local index for referring to neighbouring particles.

The *free-Lagrange method* simply uses the particles to represent the material velocity field \mathbf{U}_α and the Voronoi diagram to construct the piecewise constant cellular layer thickness $\bar{h}(\mathbf{X}_\alpha, t)$ from the principle of cell mass m_α conservation. Discretisation of Hamilton's action principle for rotating shallow water with this free-Lagrange fluid representation is the critical step in deriving a VFL method.

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3 A Variational Free-Lagrange method for 2D Shallow Water

The semi-discrete material description of Hamilton's action principle for 2d rotating shallow water in a f-plane is expressed in terms of the particle material velocities and piecewise constant cellular layer thickness.

Definition 3.1 (The Semi-Discrete Hamilton's Action Principle for Shallow Water)

$$S_d = \frac{1}{2} \int_{t_a}^{t_b} dt \sum_{\alpha} m_{\alpha} (|\mathbf{U}_{\alpha}(t)|^2 + 2\mathbf{R}_{\alpha} \cdot \mathbf{U}_{\alpha}) - V, \quad V = \frac{g}{2} \sum_{\alpha} m_{\alpha} (\bar{h}(\mathbf{X}_{\alpha}, t) + 2\bar{b}_{\alpha}). \quad (1)$$

Under the f-plane approximation, $\nabla \times \mathbf{R} = f_0 \hat{\mathbf{k}}$, $\hat{\mathbf{k}}$ is the unit vector in the direction of gravity and f_0 is the Coriolis parameter which is given by $f_0 = 2|\omega|$, where ω is the angular velocity of rotation relative to an inertial frame. g is the gravitation constant, \bar{b}_{α} is the bottom topography, assumed to be piecewise constant over cell α and V is the potential energy.

Stationarity of the discrete action principle gives the semi-discrete Euler-Lagrange particle equations

$$\dot{\mathbf{U}}_{\alpha} = \frac{g}{m_{\alpha}} \sum_i [m\bar{h}]_{\alpha}^{\beta_i} \mathbf{d}\mathbf{n}_{\alpha}^{\beta_i} - f_0 \hat{\mathbf{k}} \times \mathbf{U}_{\alpha}, \quad \dot{\mathbf{X}}_{\alpha} = \mathbf{U}_{\alpha}, \quad (2)$$

where $\mathbf{d}\mathbf{n}_{\alpha}^{\beta_i} := \hat{\mathbf{n}}^{\beta_i} \Delta l_{\alpha}^{\beta_i}$, $\Delta l_{\alpha}^{\beta_i}$ is the length of the side indexed by β_i of cell α and the operator $[\cdot]_{\alpha}^{\beta_i} = \frac{1}{2}(\cdot_{\alpha} + \cdot_{\beta_i})$ evaluates a scalar quantity at the β_i^{th} edge of cell α as the mean of that quantity over cell α and cell β_i .

By preserving symplectic structure, it follows that these semi-discrete Euler-Lagrange particle equations conserve energy. A symplectic time-stepper, such as the explicit second order Störmer-Verlet scheme, in particle position and canonical momentum co-ordinates $(\mathbf{X}_{\alpha}^n, \mathbf{P}_{\alpha}^n = m_{\alpha} \mathbf{U}_{\alpha}^n)$ at time t^n

$$[1] \quad \mathbf{P}_{\alpha}^{n+\frac{1}{2}} = \mathbf{P}_{\alpha}^n - \frac{\Delta t}{2} V_{\mathbf{X}}(\mathbf{X}_{\alpha}^n), [2] \quad \mathbf{X}_{\alpha}^{n+1} = \mathbf{X}_{\alpha}^n + \Delta t \frac{\mathbf{P}_{\alpha}^{n+\frac{1}{2}}}{m_{\alpha}}, [3] \quad \mathbf{P}_{\alpha}^{n+1} = \mathbf{P}_{\alpha}^{n+\frac{1}{2}} - \frac{\Delta t}{2} V_{\mathbf{X}}(\mathbf{X}_{\alpha}^{n+1}),$$

preserves the symplectic two-form $\omega^n = \sum_{\alpha=1}^{N^2} d\mathbf{X}_{\alpha}^n \wedge d\mathbf{P}_{\alpha}^n$, where $V_{\mathbf{X}}$ denotes the gradient of the potential energy. Consequently, this time-stepper will ensure the absence of secular drift in the energy error of the fully discrete equations over long simulations as the graph in Figure 2 shows.

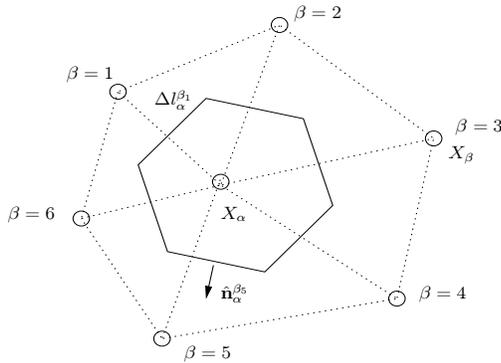


Fig. 1 A hexagonal ($n_e = 6$) Voronoi cell containing the particle with label α . Each cell edge is of length $\Delta l_{\alpha}^{\beta_i}$ and labelled by β_i .

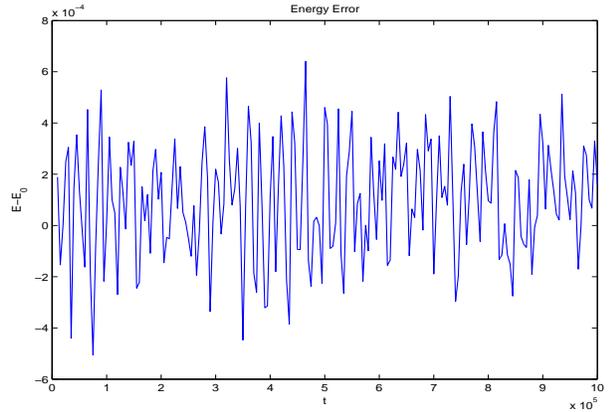


Fig. 2 The graph shows the energy error of a 1D simulation over one million time steps of size $\Delta t = 0.01$ using 128 cells. The shallow water covers a periodic domain $[0, 2\pi)$ rotating with $f_0 = 2\pi$ and is initialized with a gaussian perturbation of the layer thickness. The error exhibits no secular drift in energy.

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