Various forms of F=ma:

Building a discrete model of force balance that does what we want it to do.

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Office of Science U.S. DEPARTMENT OF ENERGY

Motivation

The discrete form F=ma has strong implications on the representation of energetics and vortex dynamics.

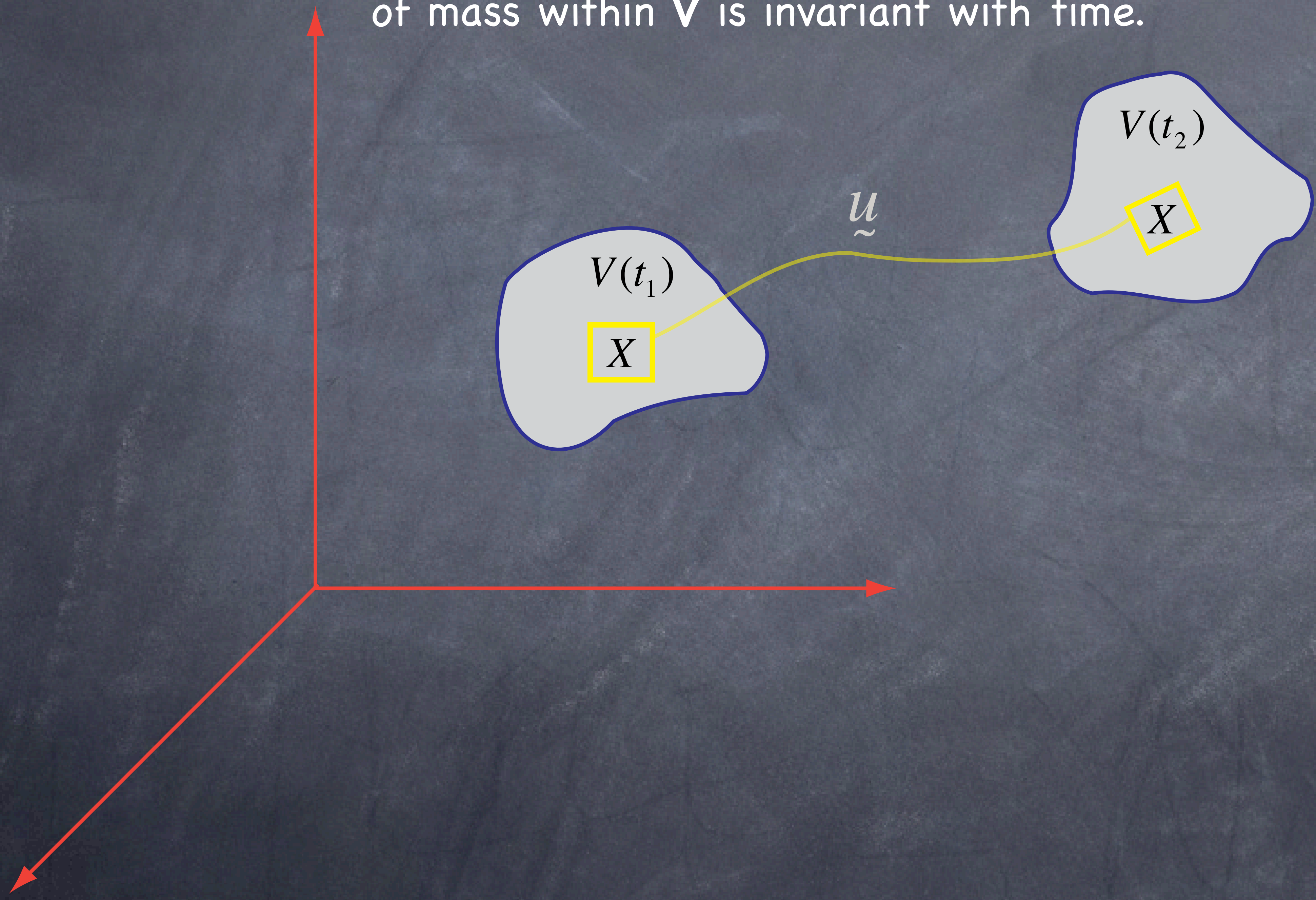
Getting F=ma "correct" is a prerequisite to a robust model of geophysical fluid dynamics.

Outline

- Mass, Momentum and Circulation developed in a Lagrangian reference frame.
- The four common ways to express F=ma (advective, conservative, invariant, vor/div -- each to be defined) and their respective advantages and disadvantages.
- Vortex dynamics and F=ma in a discrete system, constructing a discrete analog to Kelvin's Circulation Theorem.

Our Lagrangian reference frame defined.

Define a volume of fluid (**V**) composed of a set of particles (**X**) enclosed (at all times) by a surface (**S**). By construction, the amount of mass within **V** is invariant with time.



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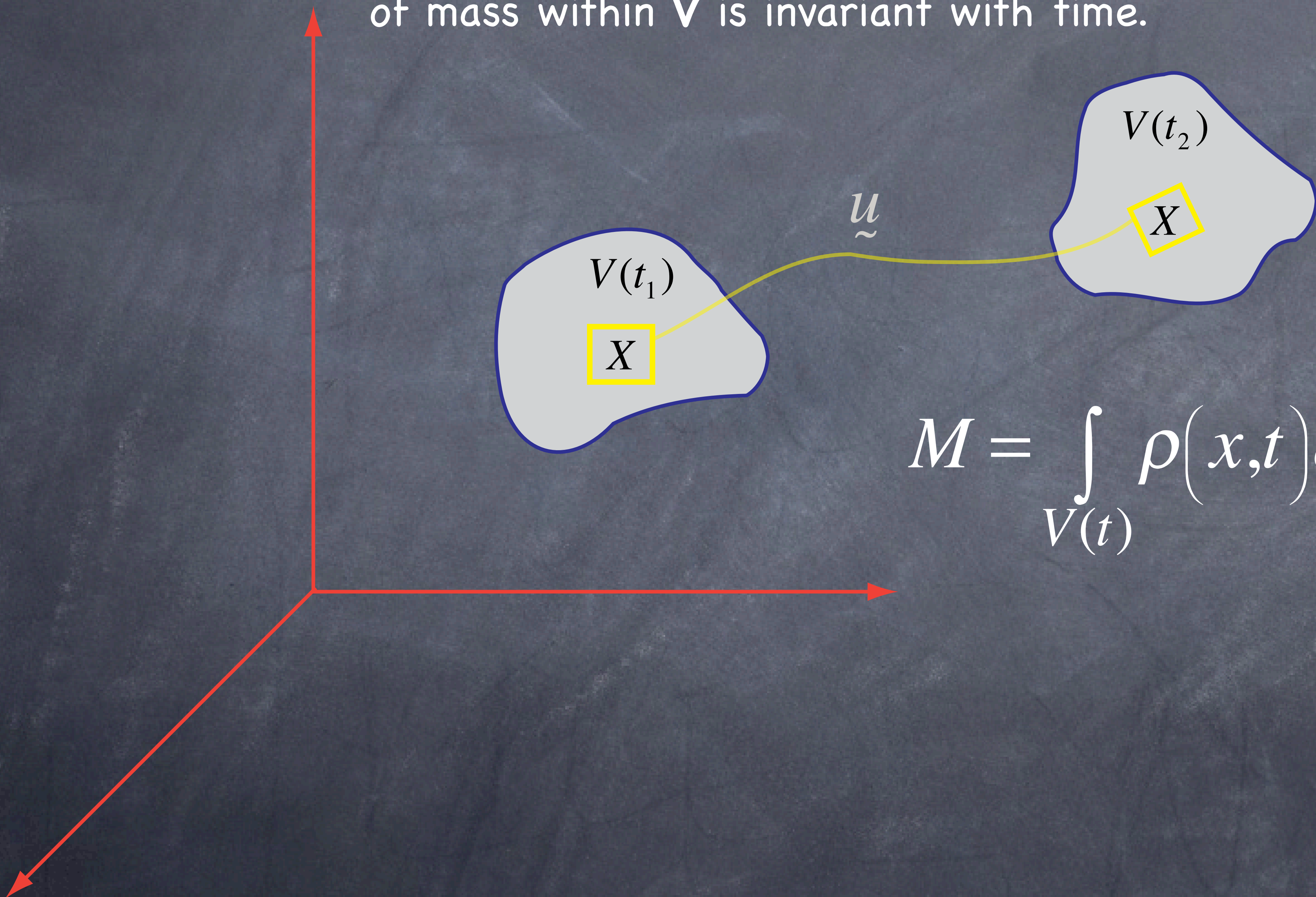

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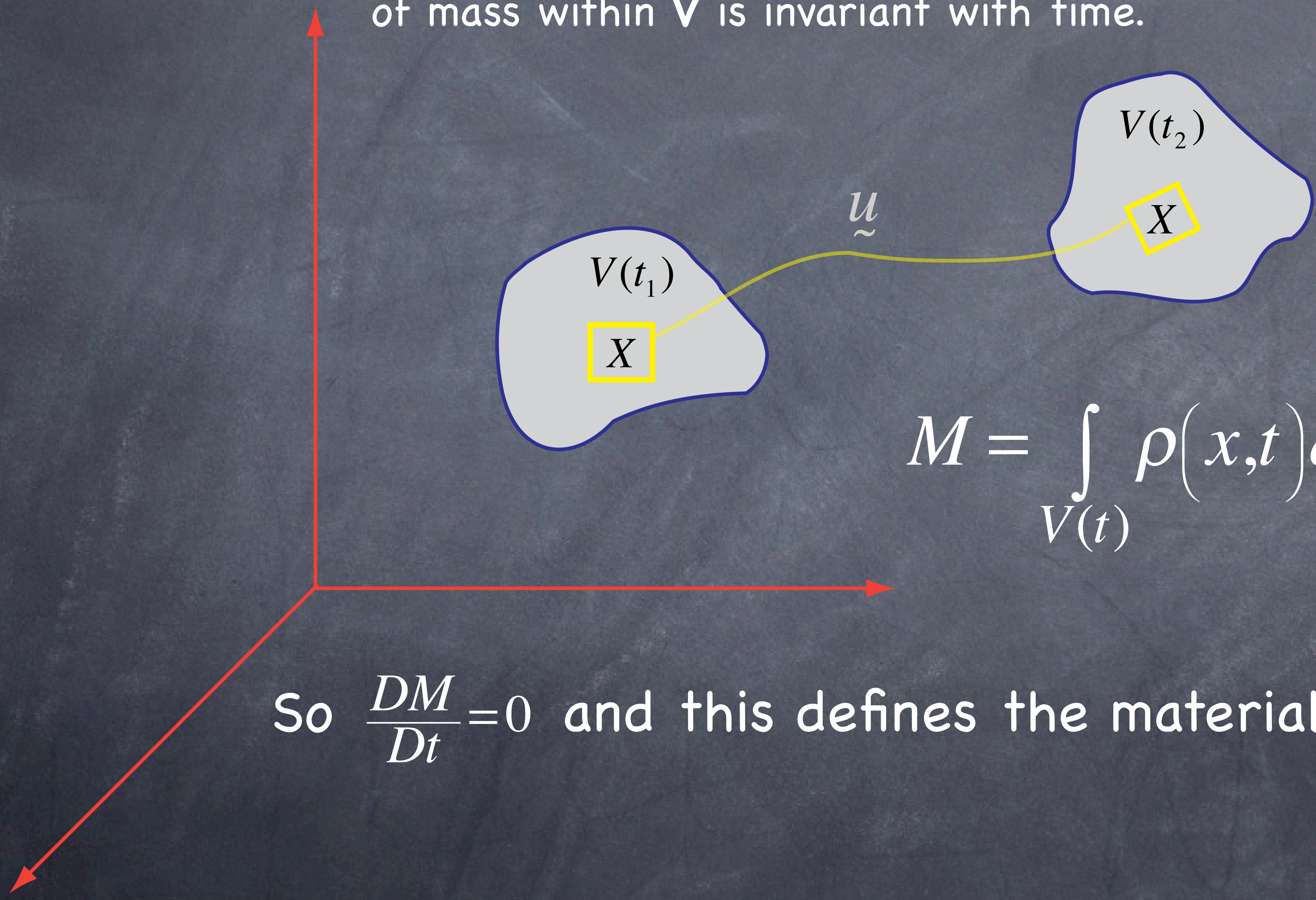

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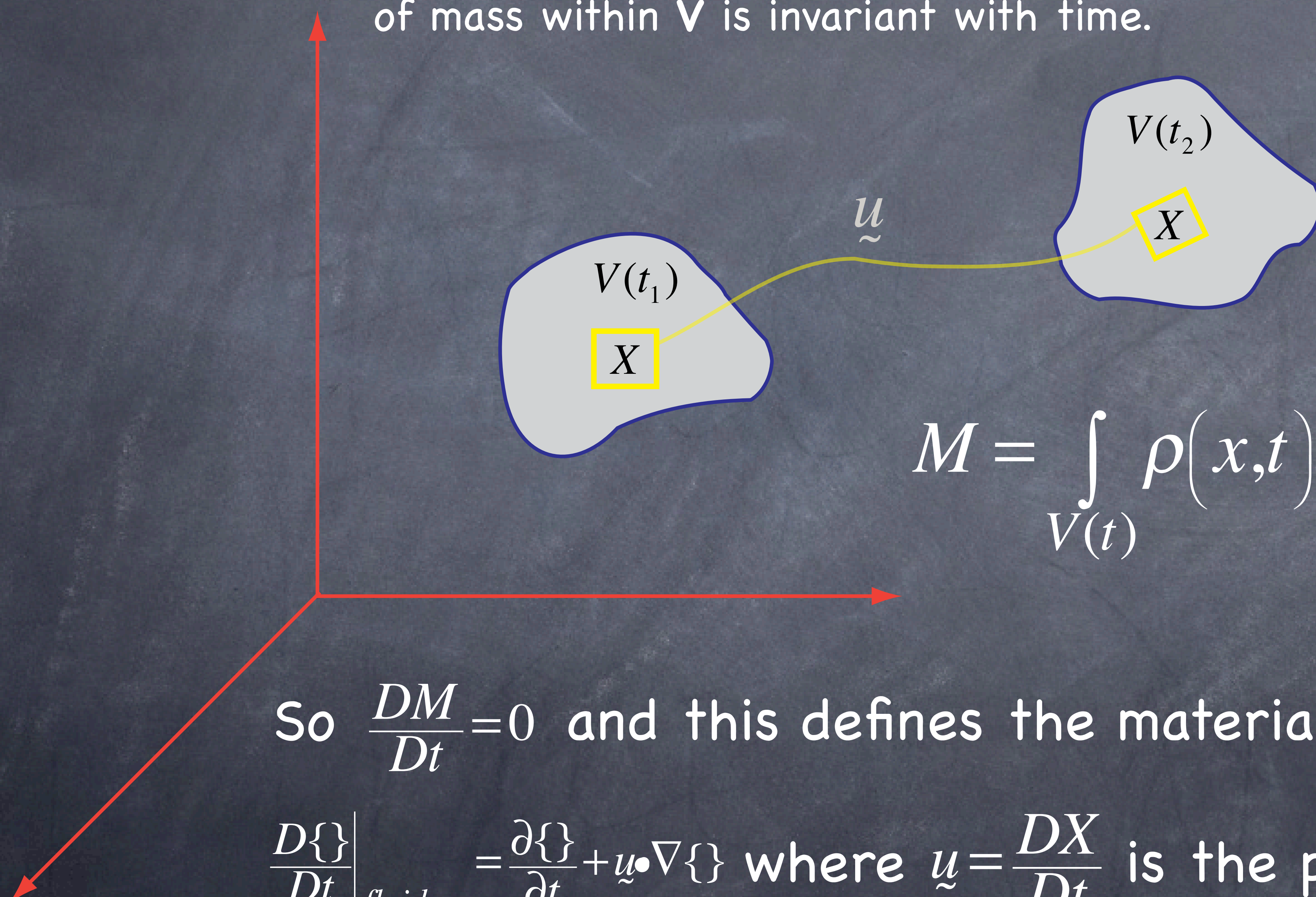

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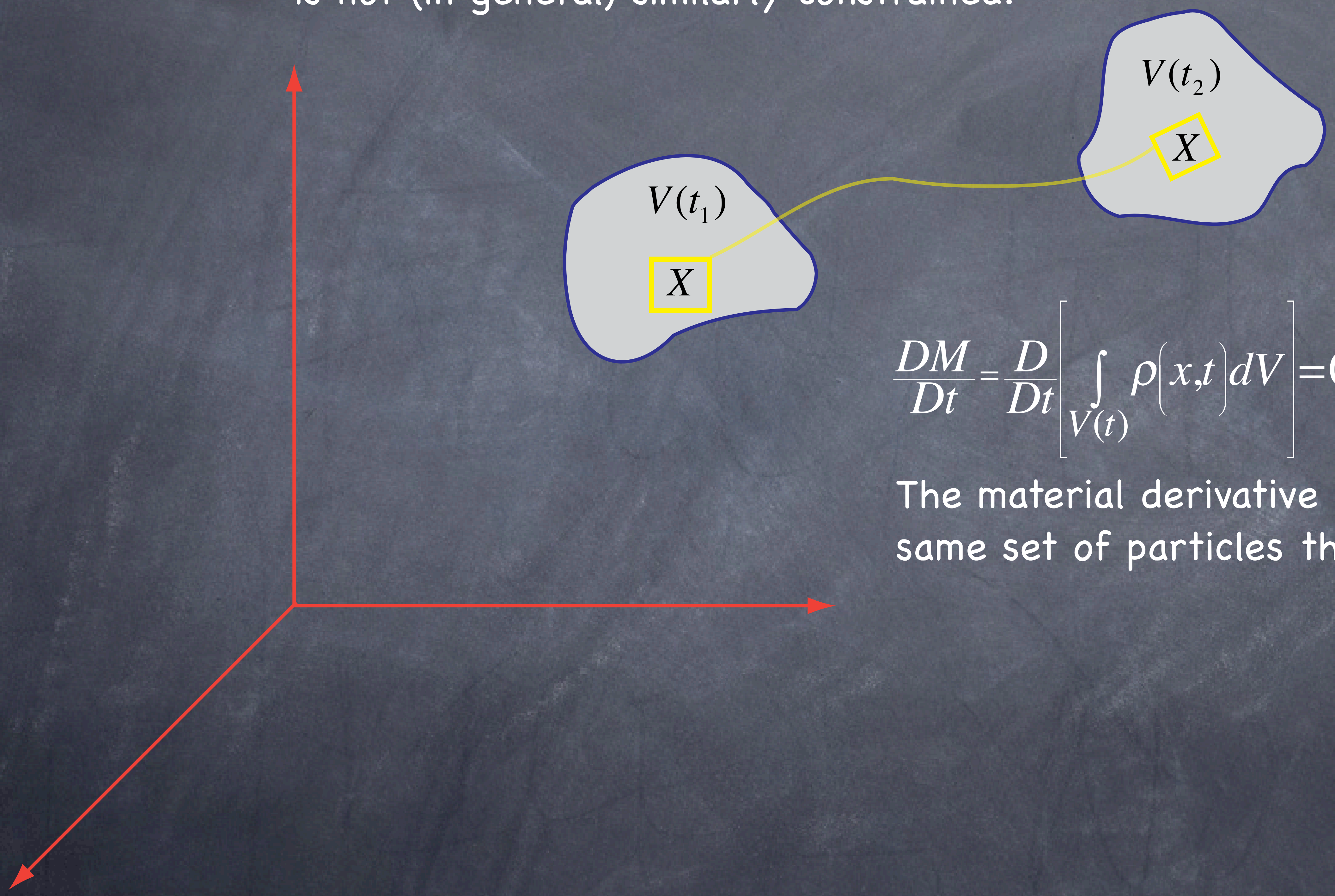
$\left.\dfrac{D\{\}}{Dt}\right|_{\substack{fluid\\particle}} = \dfrac{\partial\{\}}{\partial t} + \underset{\sim}{u}\bullet\nabla\{\}$ where $\underset{\sim}{u} = \dfrac{DX}{Dt}$ is the particle velocity.

Conservation of Mass

While mass is constant within our Lagrangian control volume, the volume is not (in general) similarly constrained.



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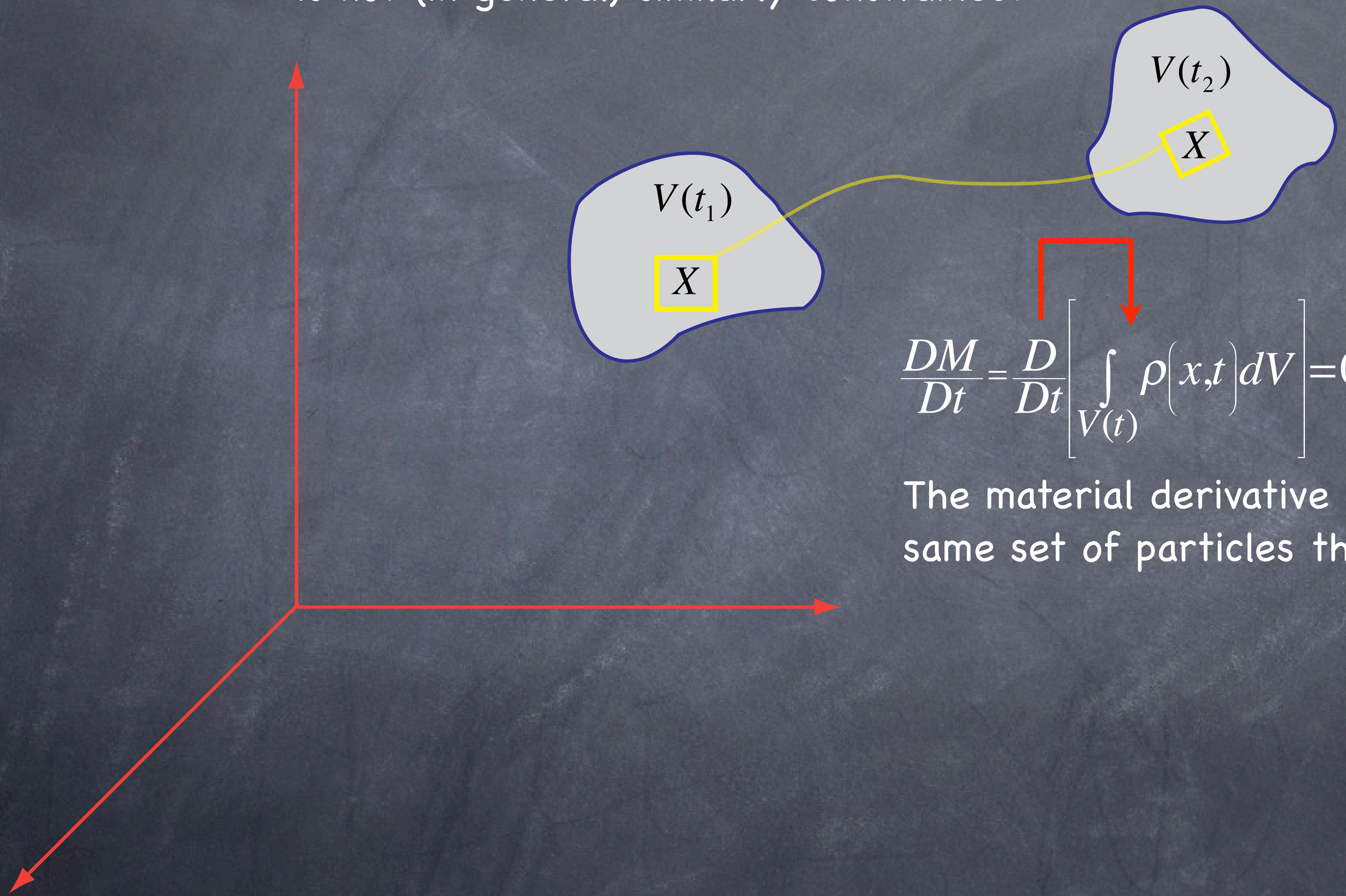

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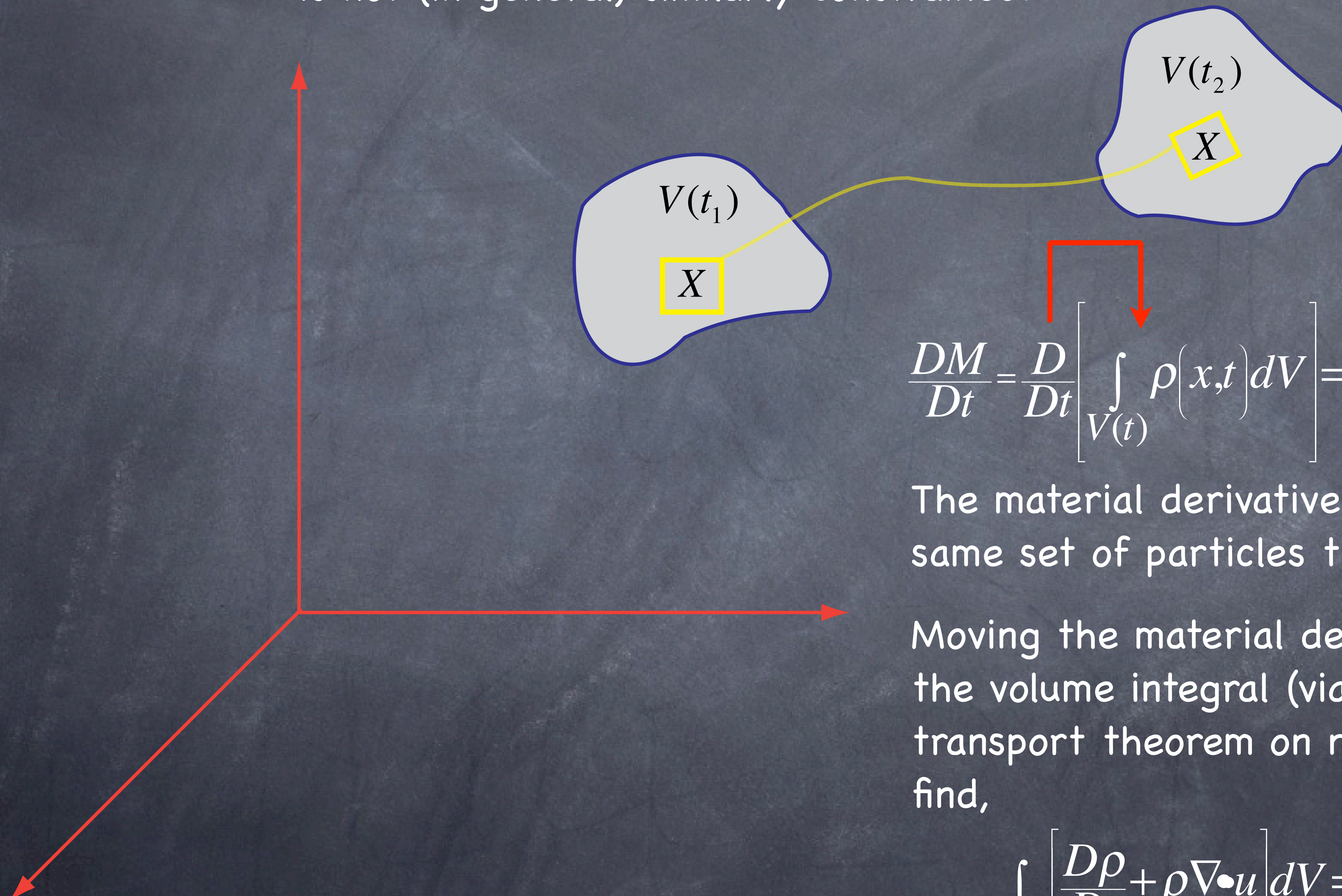

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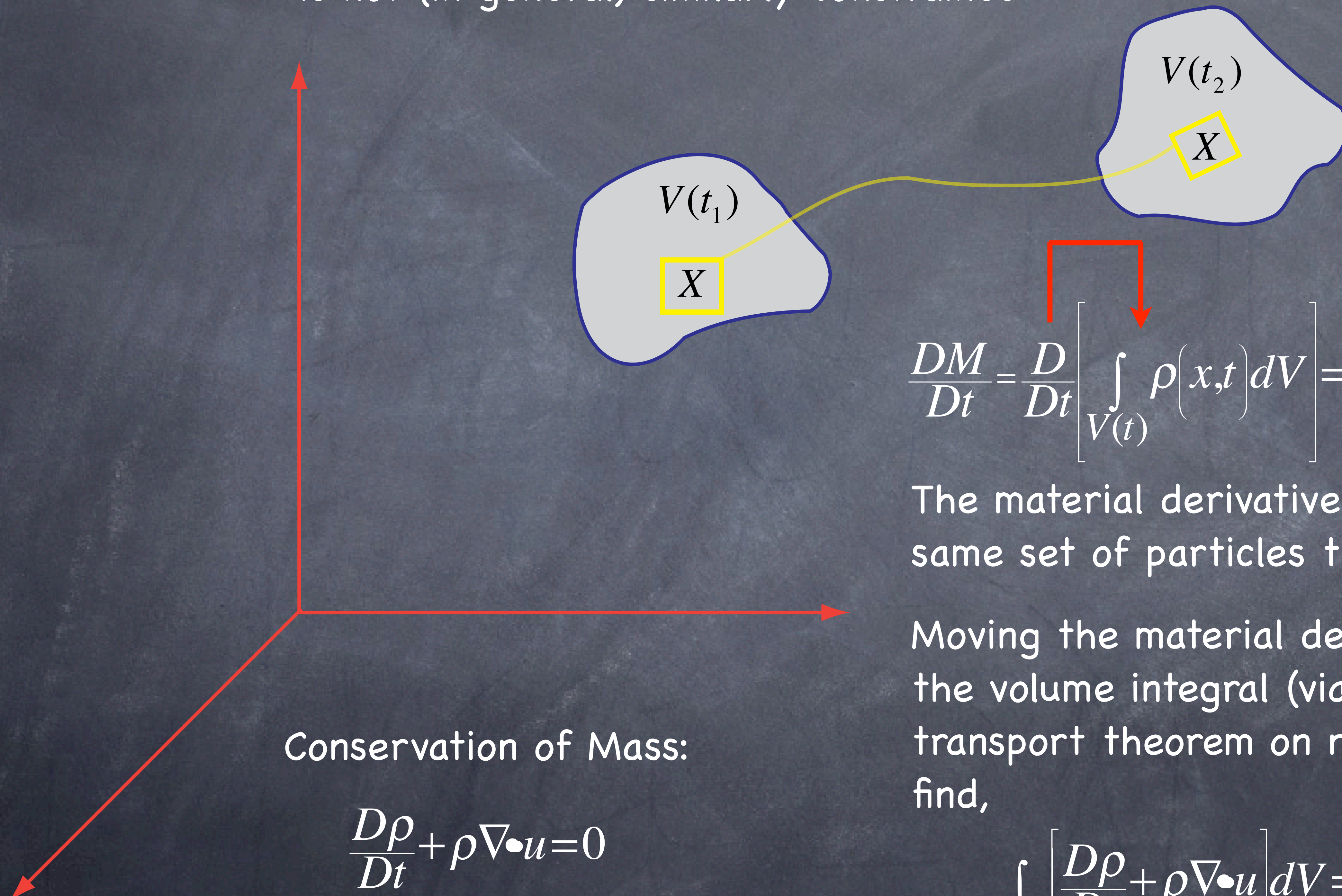

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Reynold's Transport Theorem

a foundation of finite-volume methods

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The time rate of change of a quantity moving with the Lagrangian velocity defined within a volume V.

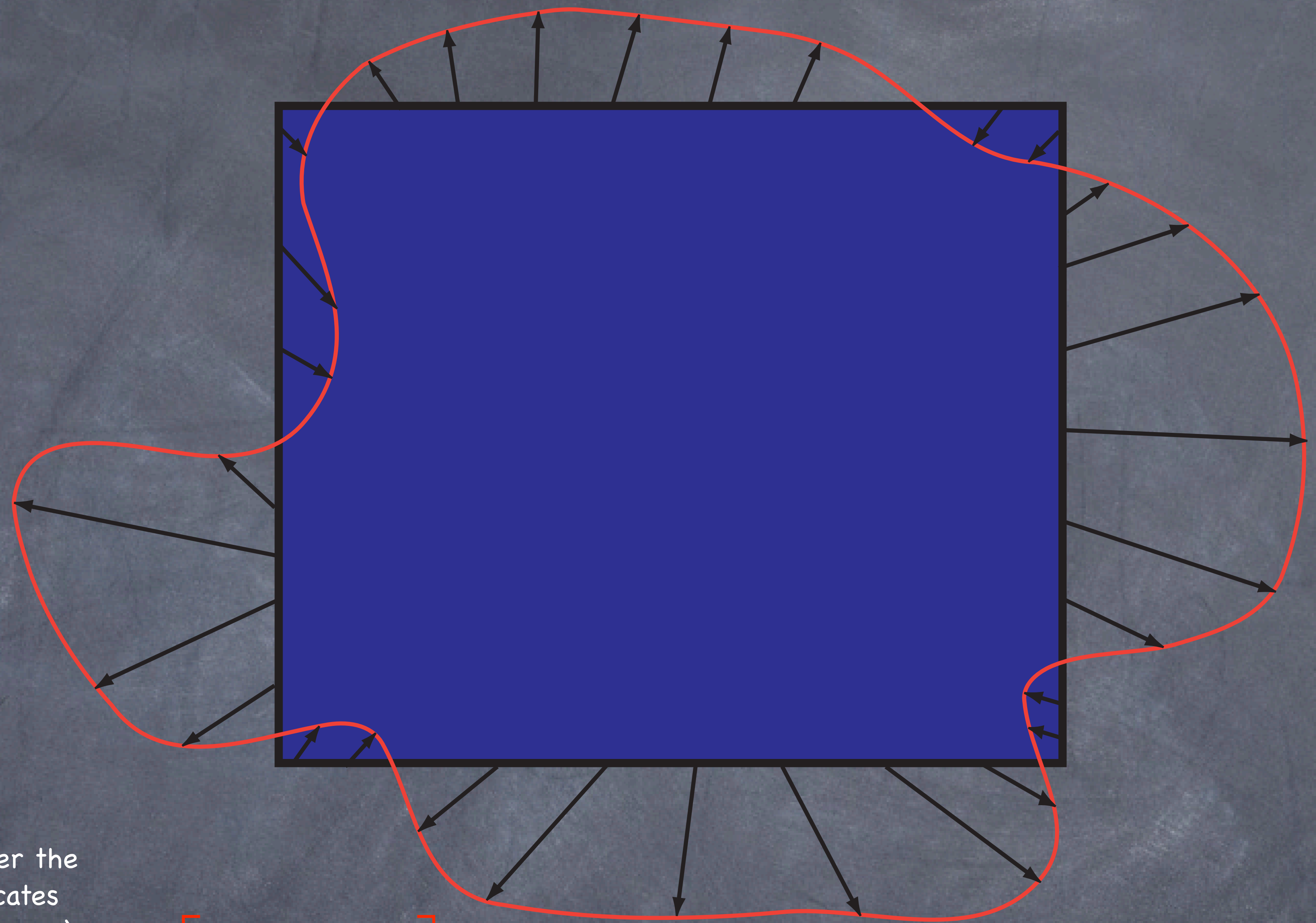
=

The instantaneous time rate of change of that quantity within the volume.

+

The net flux of that quantity across the surface bounding the volume.

RTT and Control Volumes



(note: integration is over over the blue domain. red line indicates how the square domain deforms.)

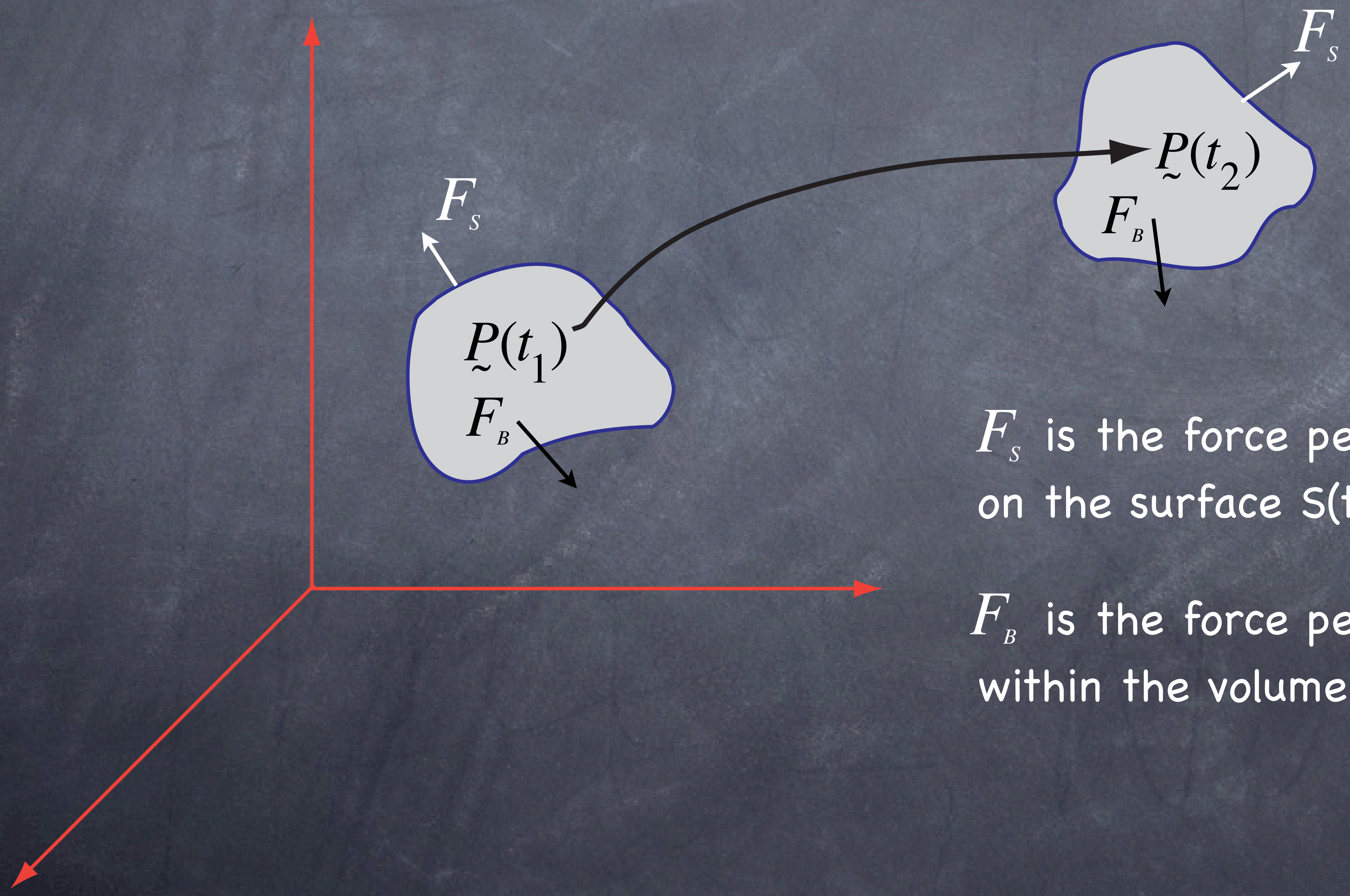
$$\frac{D}{Dt}\left[\int_V \rho \, dV\right] = \int_V \frac{\partial \rho}{\partial t} dV + \int_S \rho \underset{\sim}{u} \bullet \underset{\sim}{n} \, dS = 0$$

This allows us to recast statements most naturally expressed in a Lagrangian reference frame in an Eulerian (fixed) reference frame.

Our statement of F=ma for a material volume

The extrinsic quantity is momentum $\underset{\sim}{P} = \int_{V(t)} \rho \underset{\sim}{u}\, dV$ and the statement of Newton's 2nd law for the volume of fluid is,

$$\frac{D\underset{\sim}{P}}{Dt} = \frac{D}{Dt}\left[\int_{V(t)} \rho \underset{\sim}{u}\, dV\right] = \int_{V(t)} F_B\, dV + \int_{S(t)} F_S\, dS$$



F_S is the force per unit area acting on the surface S(t)

F_B is the force per unit volume acting within the volume V(t)

RTT (and mass conservation) applied to F=ma

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Taking the limit of V going to zero, a point-wise perspective of F=ma.

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A control volume perspective of F=ma

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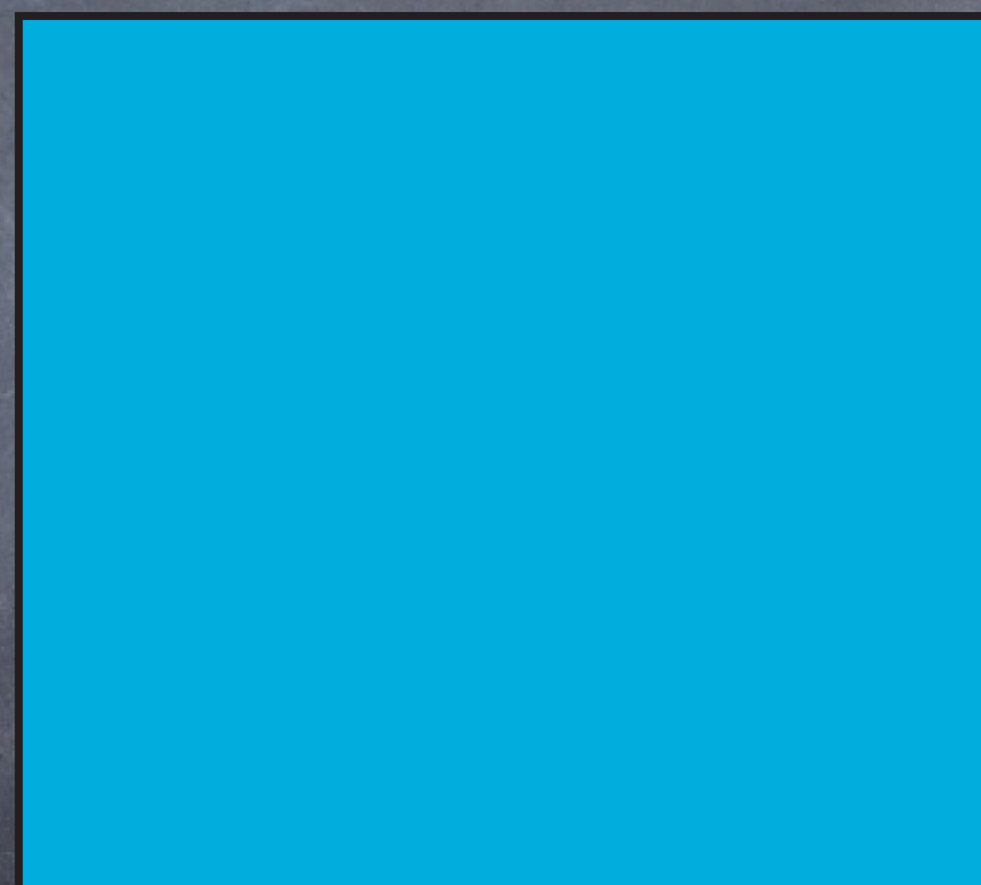

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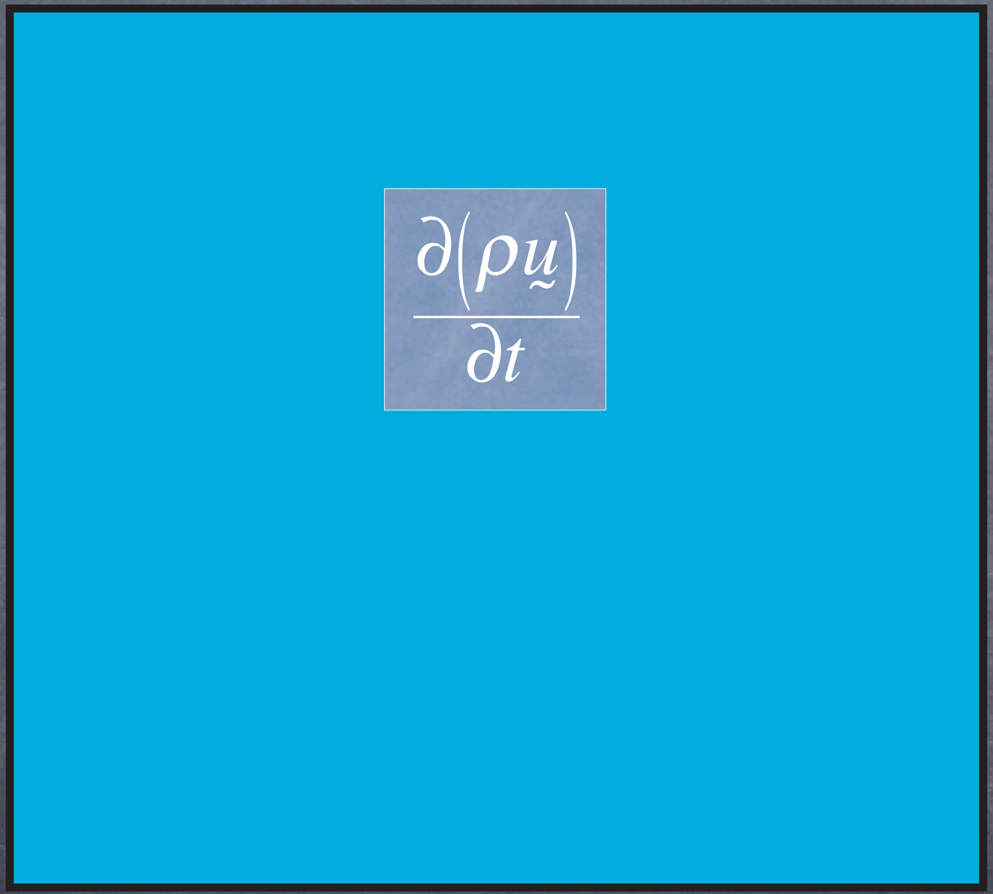

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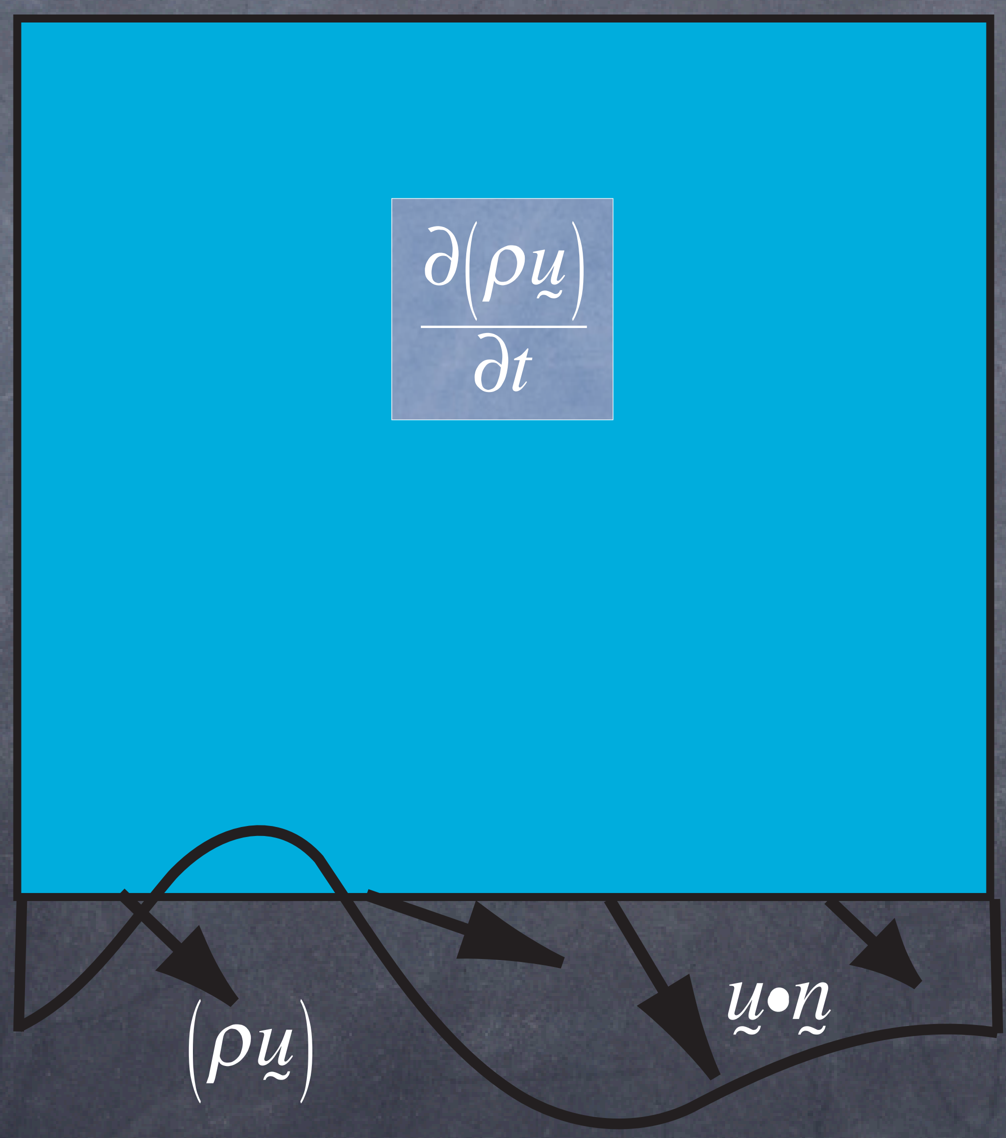

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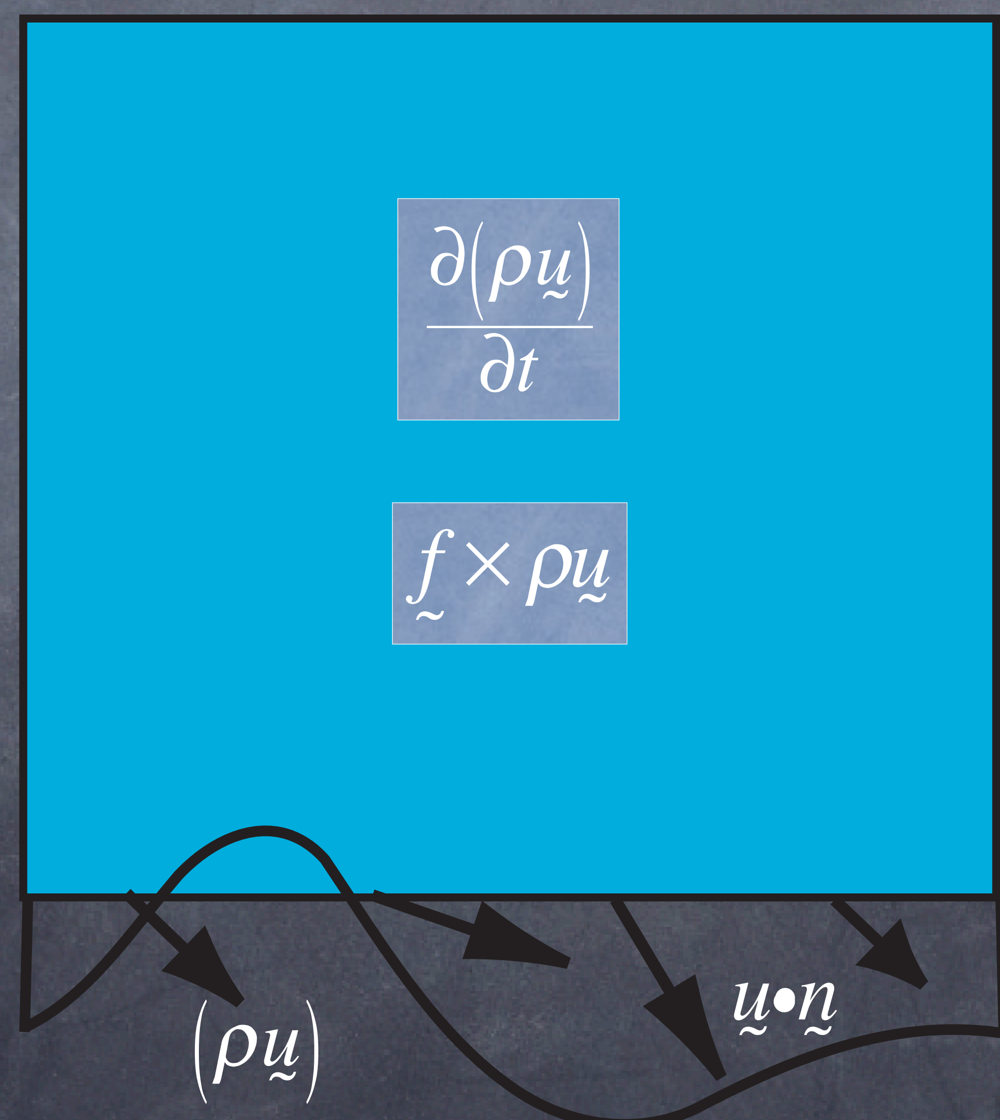

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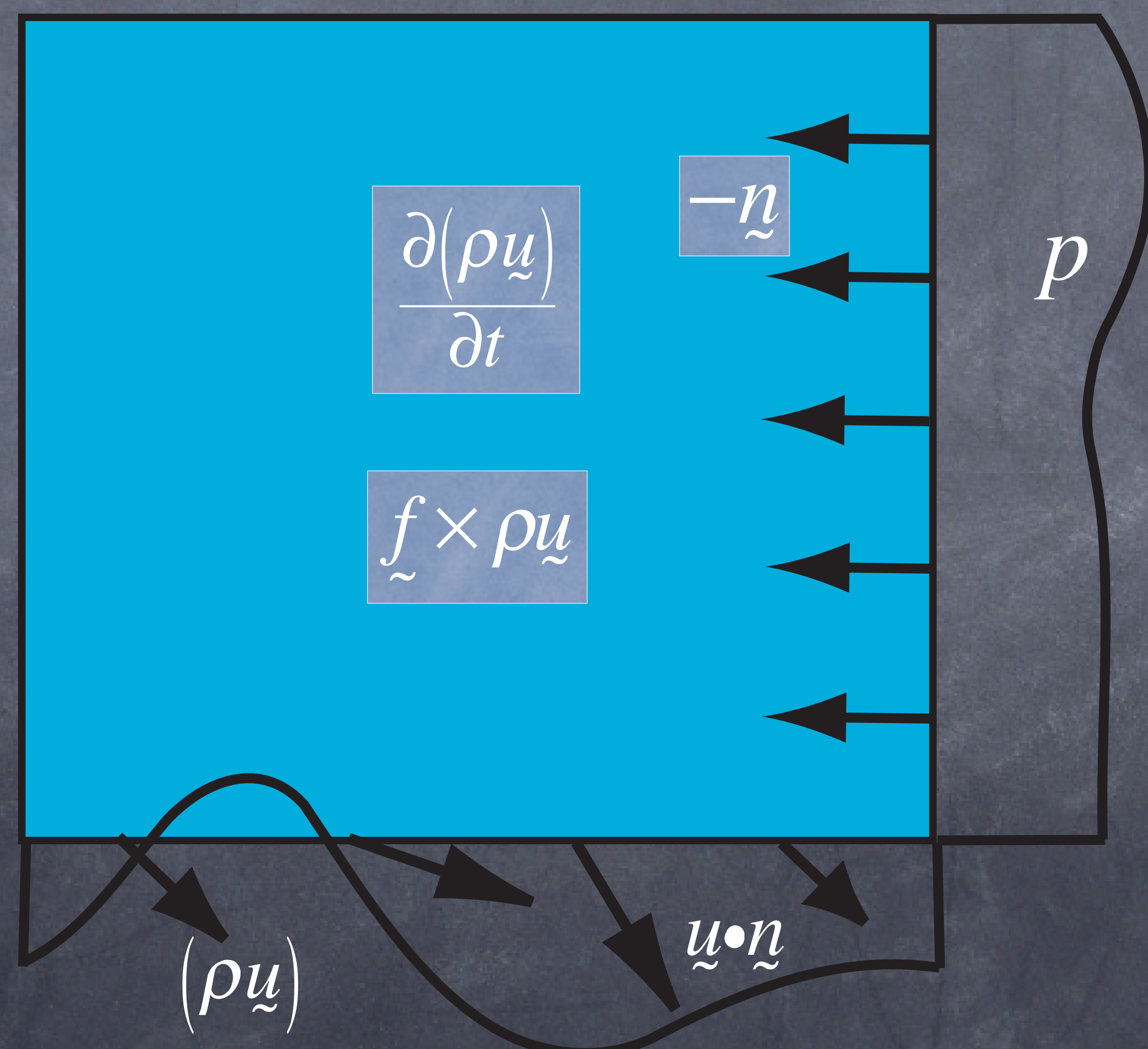

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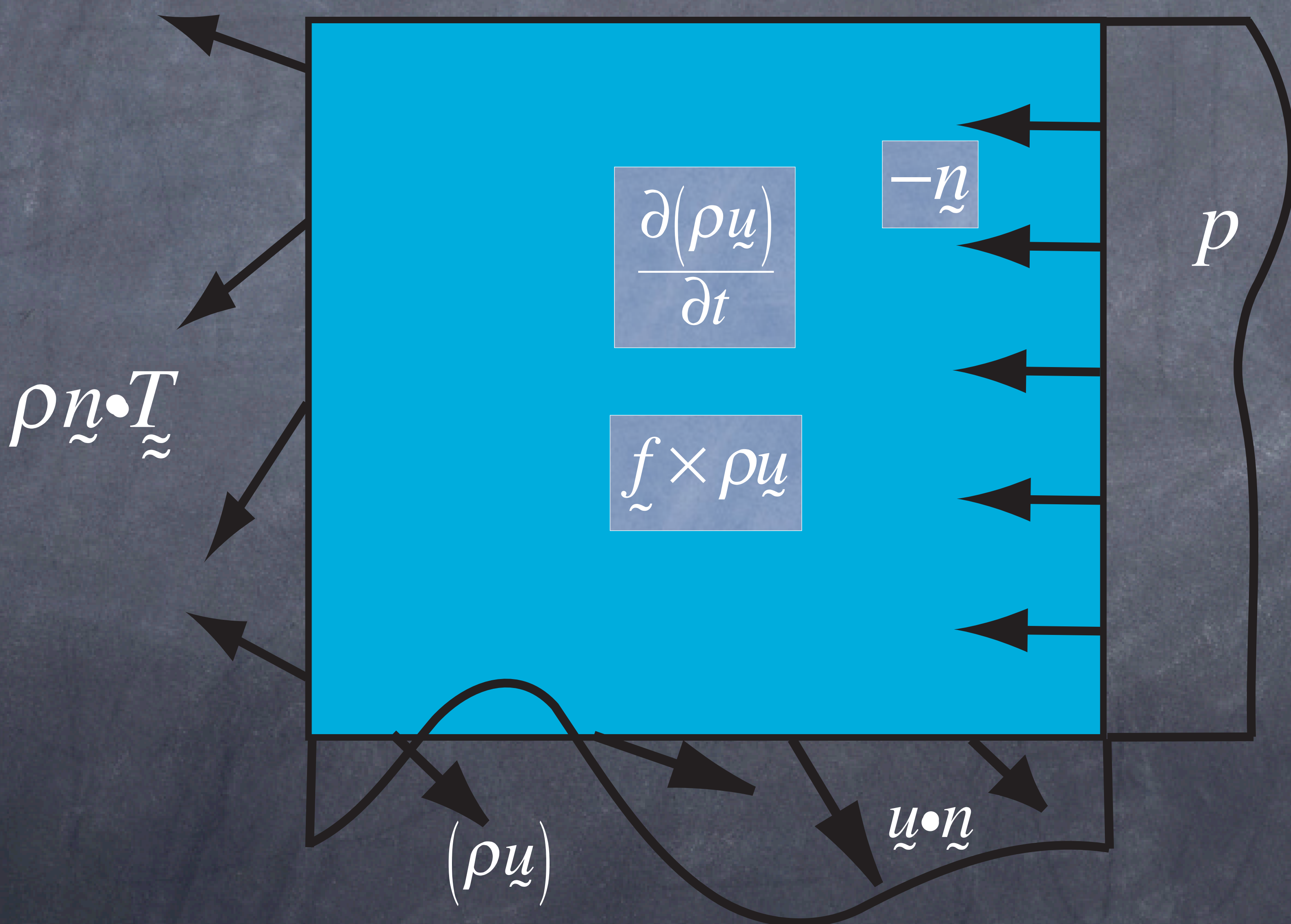
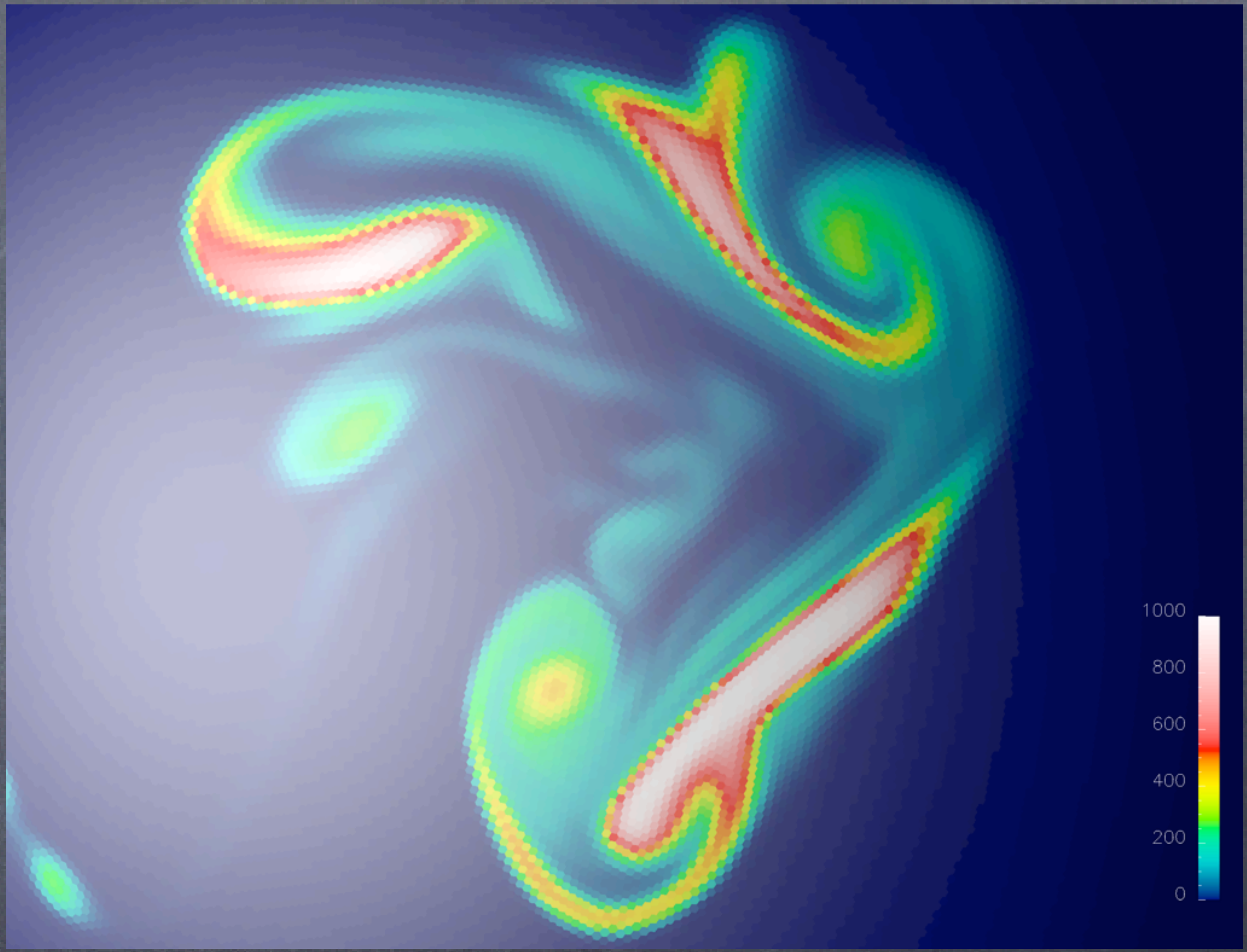


One aspect of F=ma that is of particular interest is the implied vorticity/circulation/angular momentum budget.

Vorticity field at day 10
Shallow-water equation.



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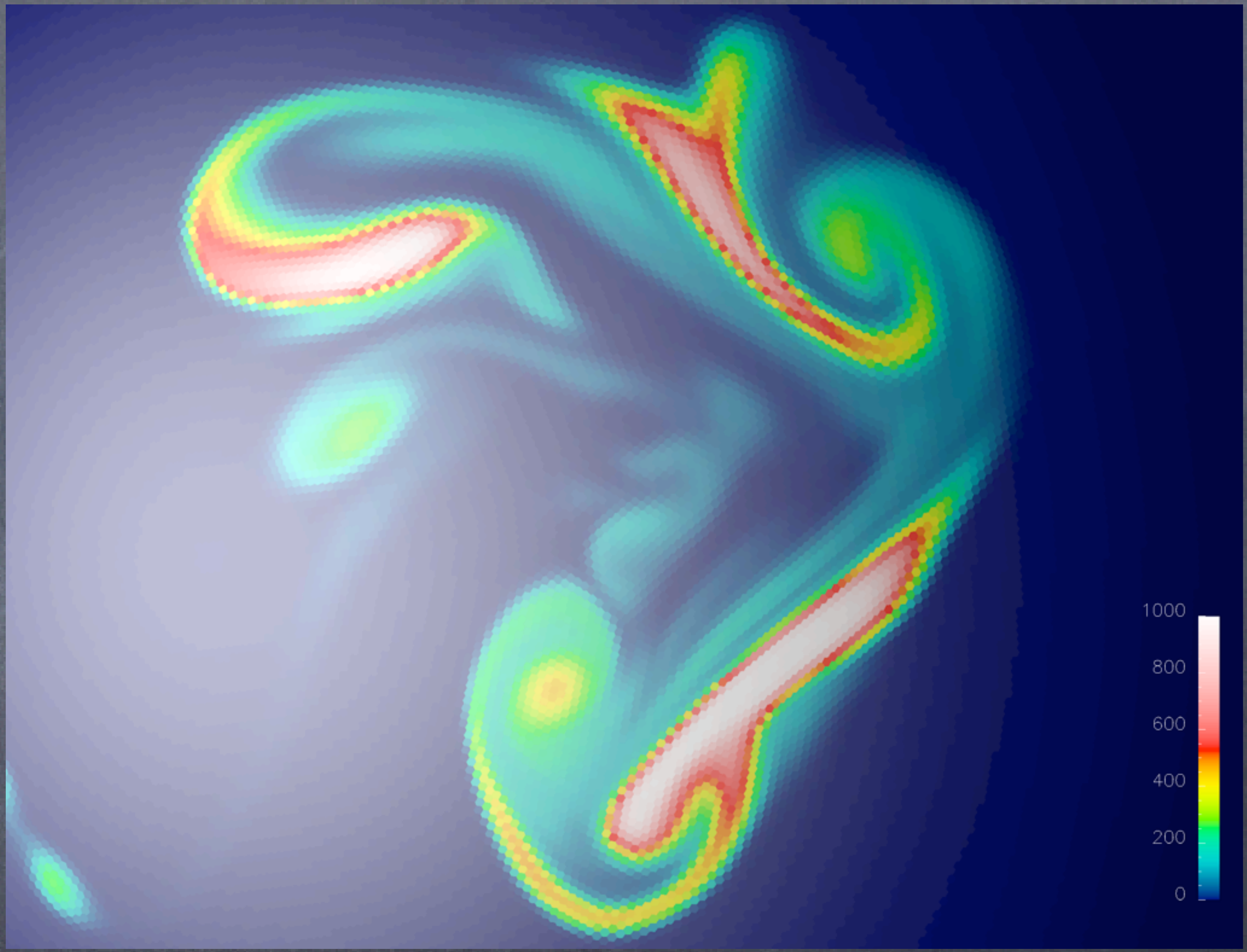

Why?

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2) Strongly influences long-time dynamics.

3) Explains a significant fraction of velocity field.

4) Critical for robustness of numerical models.

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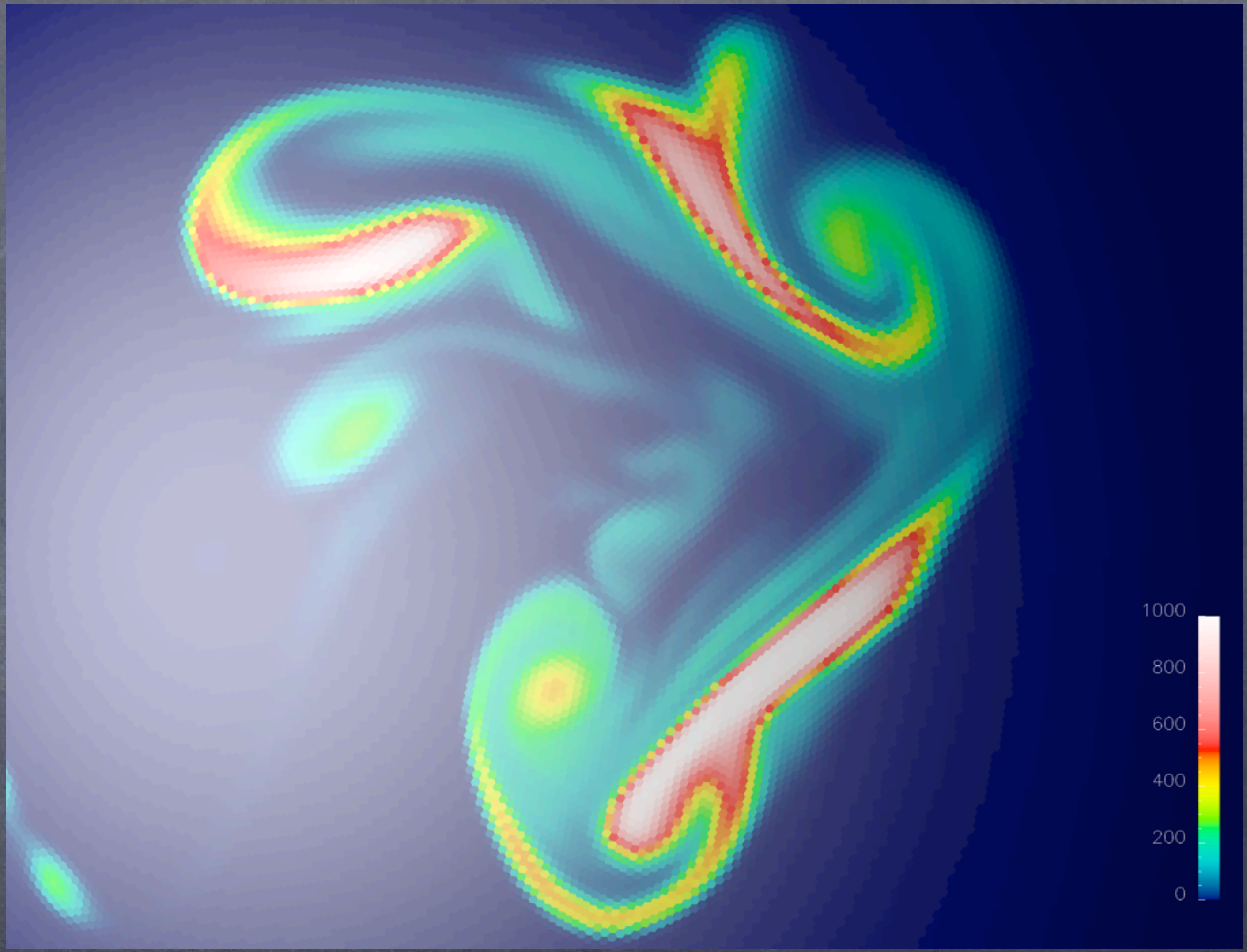

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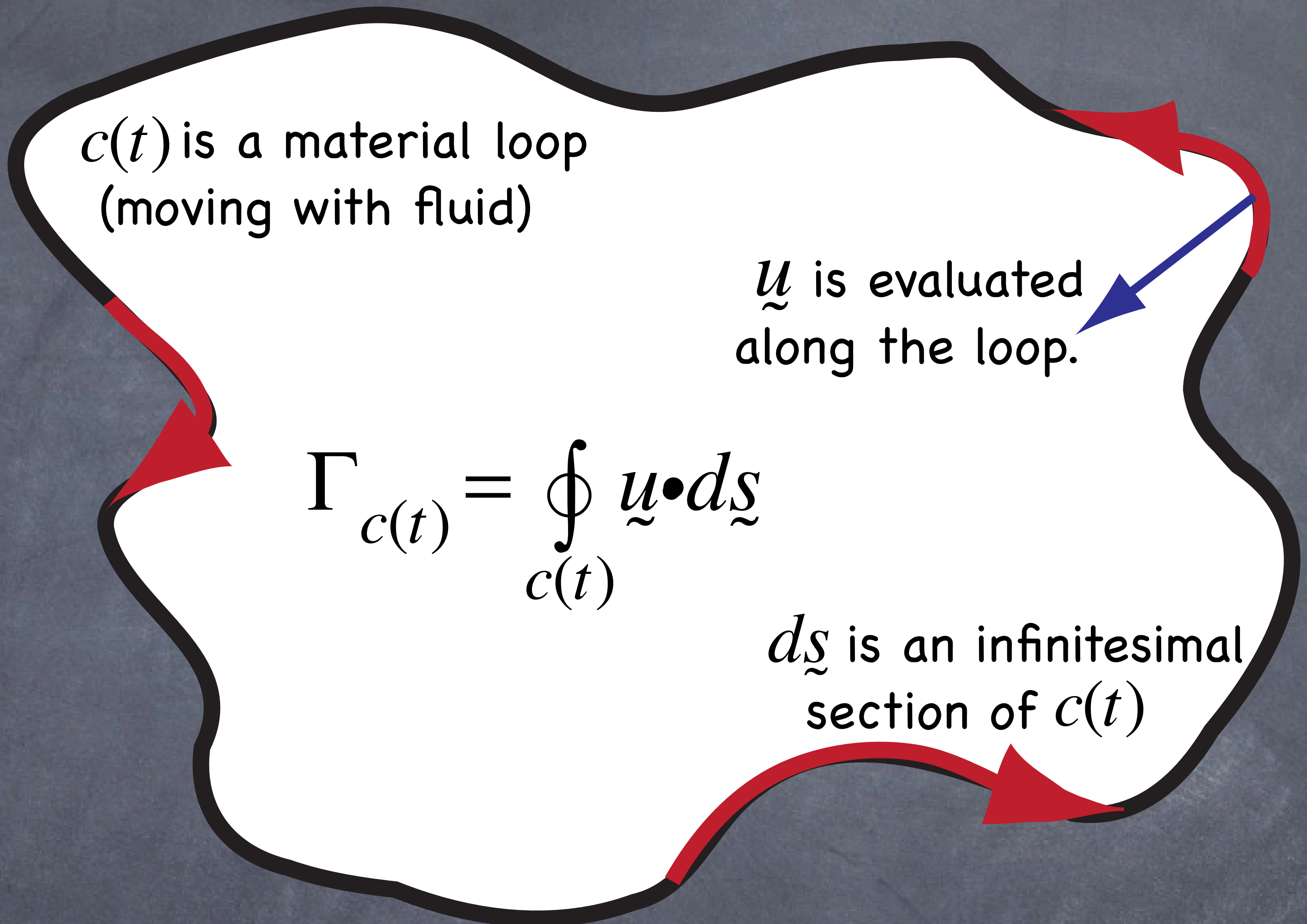
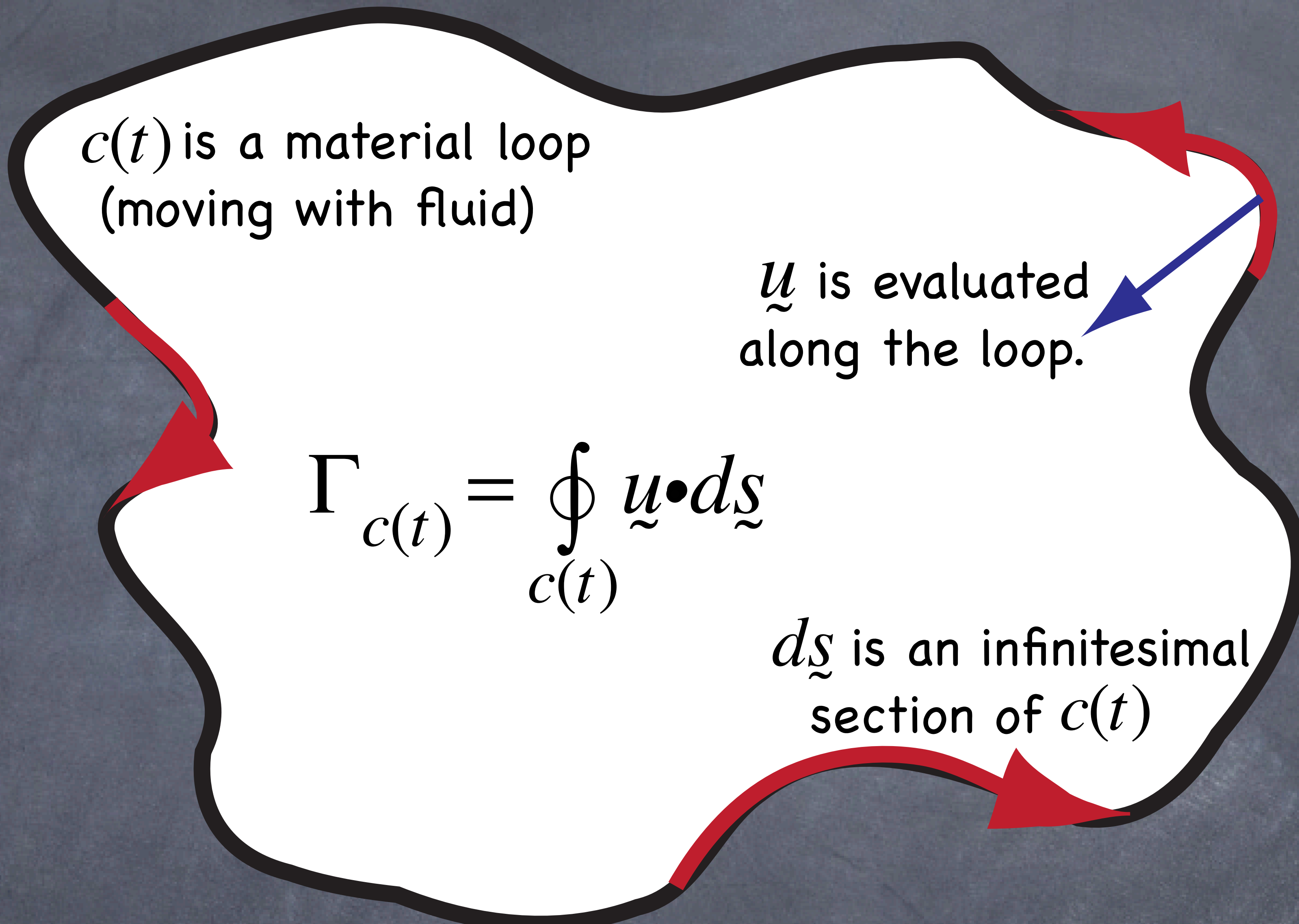
For clarity, let's limit our analysis to flow in the tangent plane, i.e. "horizontal" flow and the vertical component of vorticity. Nothing precludes the extension to full 3D flows.

Circulation defined ....



$$\Gamma_{c(t)} = \oint_{c(t)} \underset{\sim}{u} \bullet d\underset{\sim}{s}$$

Circulation defined ....



$\Gamma_{c(t)} = \oint_{c(t)} \underset{\sim}{u} \bullet d\underset{\sim}{s} = \bar{\omega} dA$, where $\bar{\omega}$ is the area-mean vorticity.

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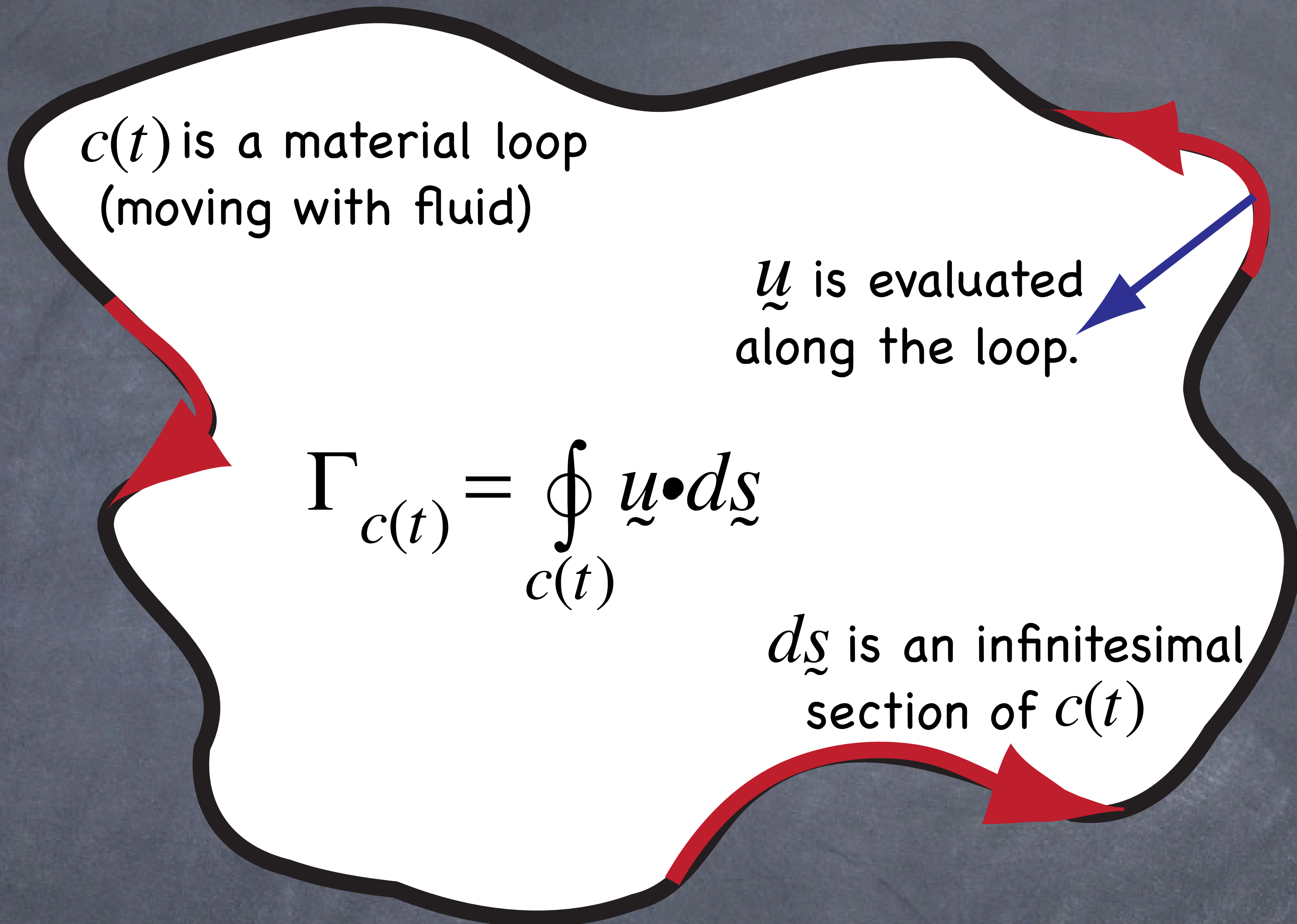

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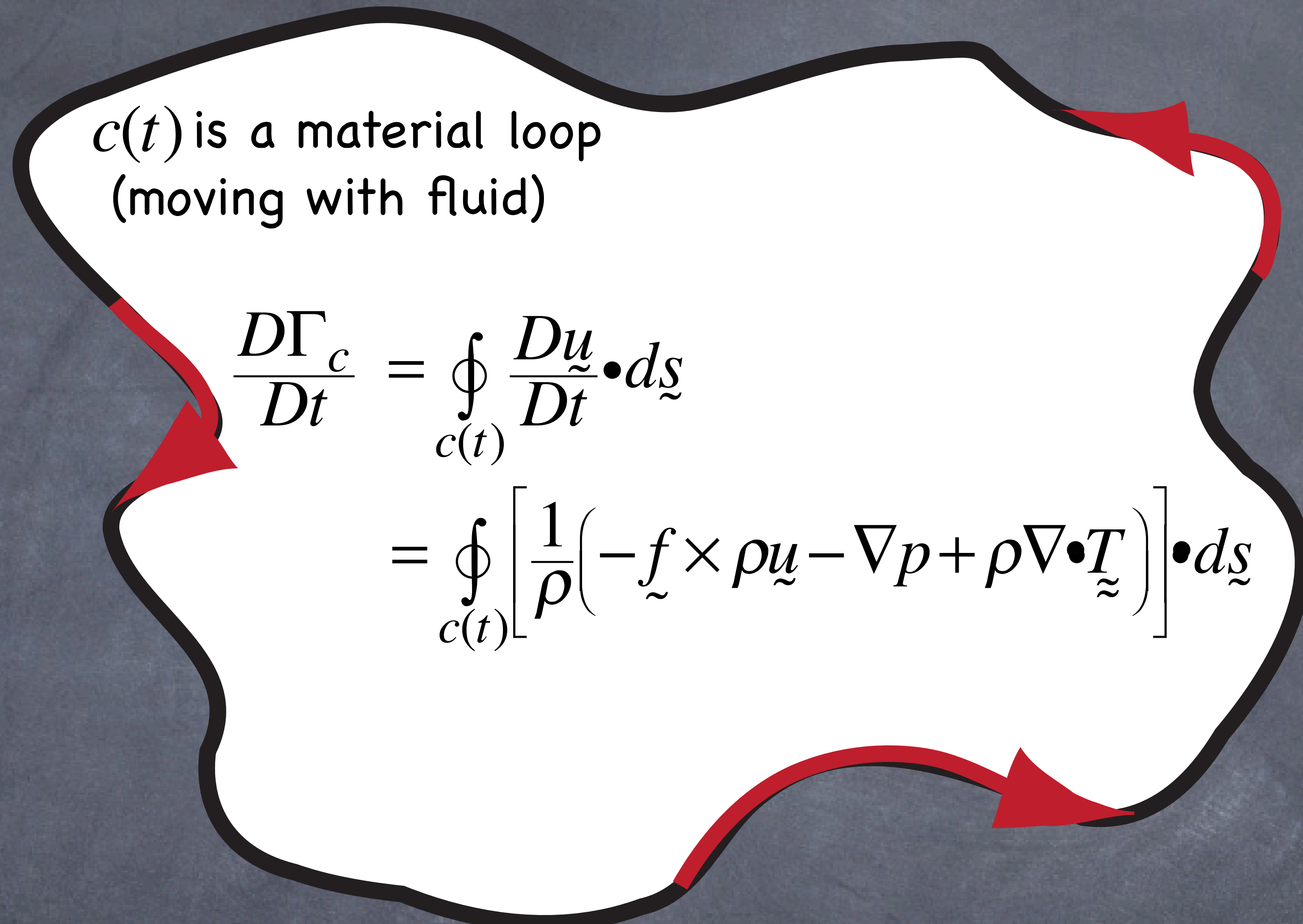
Evolution of Circulation



$$\frac{D\Gamma_c}{Dt} = \oint\limits_{c(t)} \frac{D\underset{\sim}{u}}{Dt} \bullet d\underset{\sim}{s}$$

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Evolution of Circulation

$c(t)$ is a material loop
(moving with fluid)

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$\nabla \bullet \underset{\approx}{T} = 0$ (ignore viscosity)

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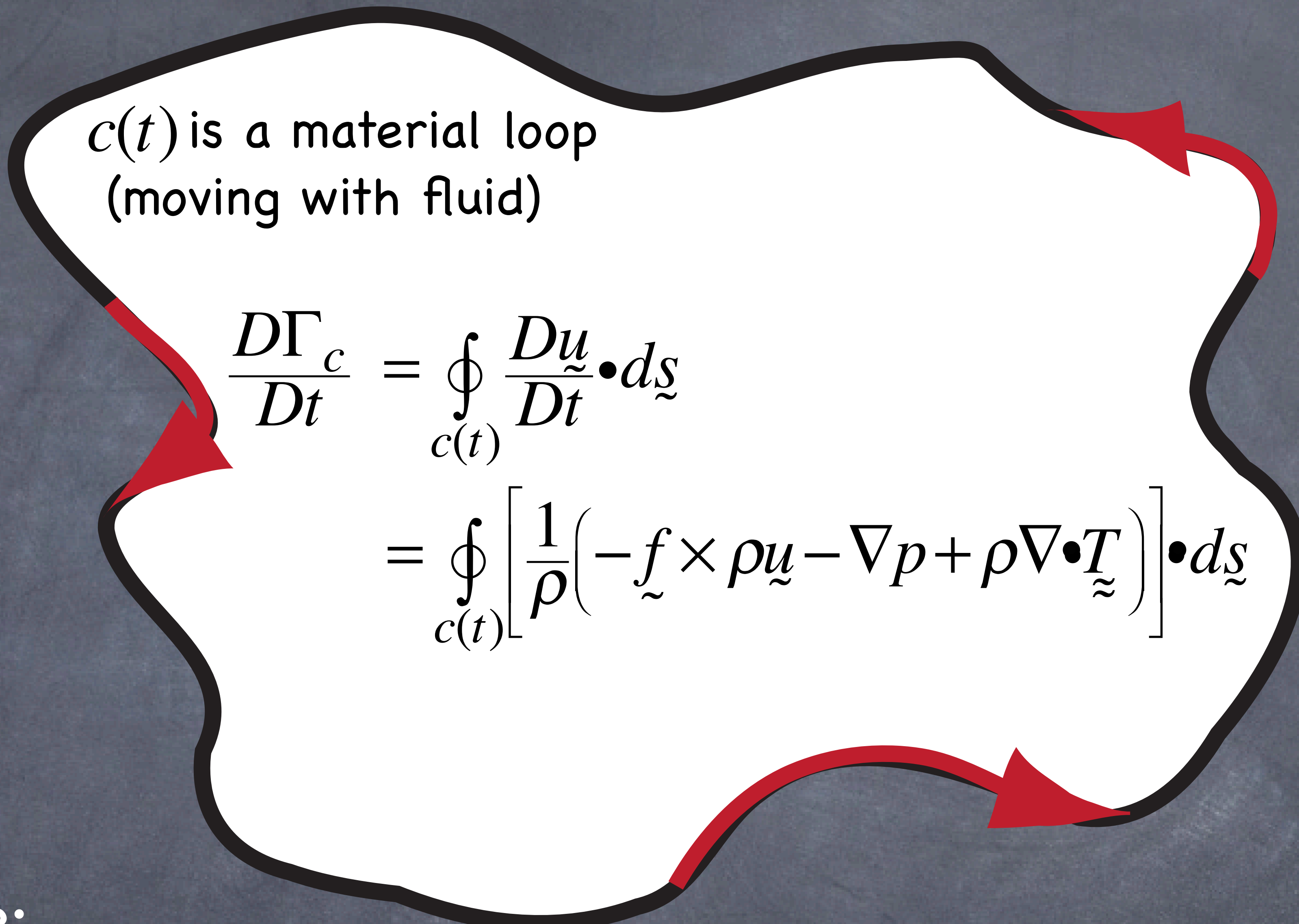

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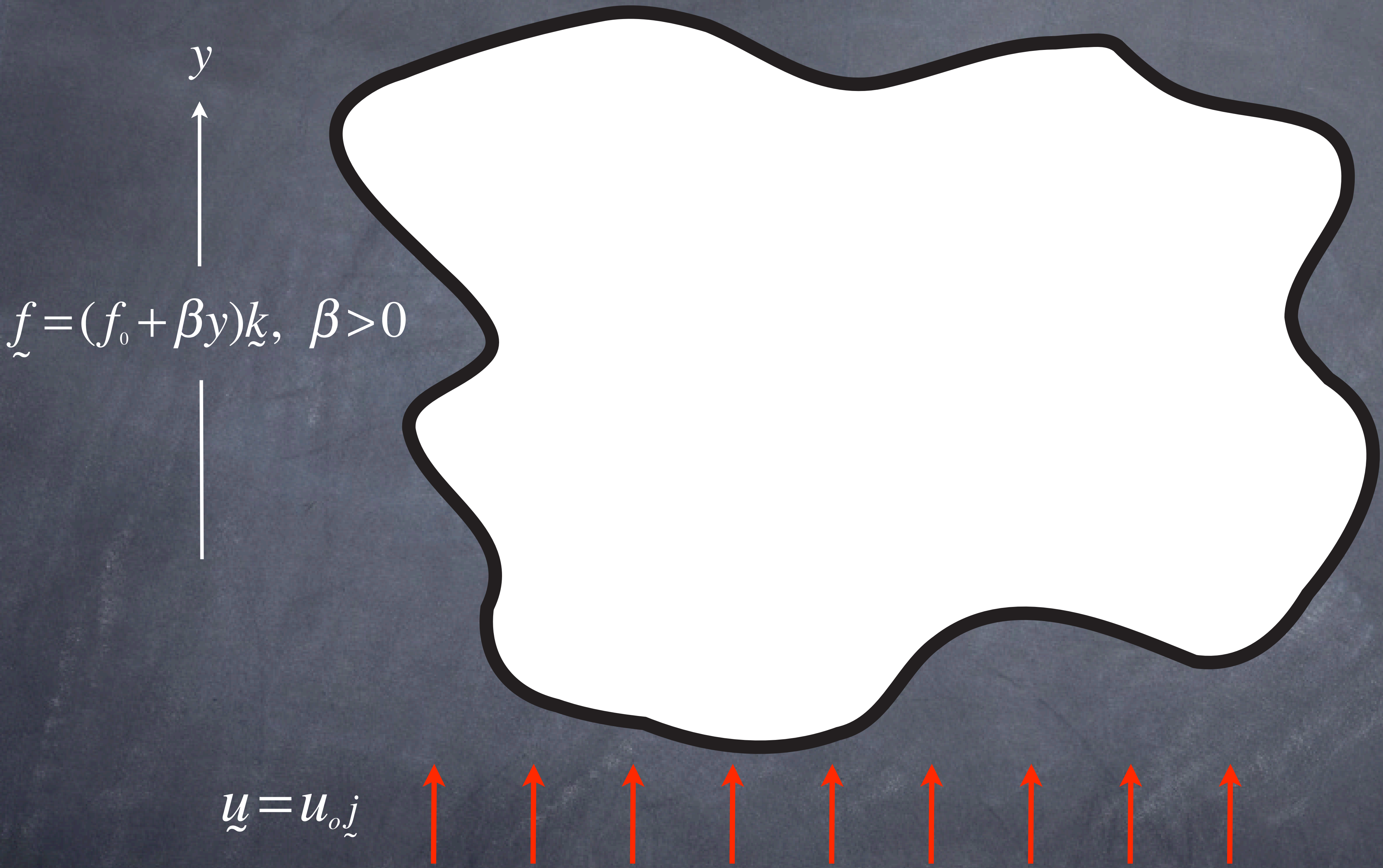

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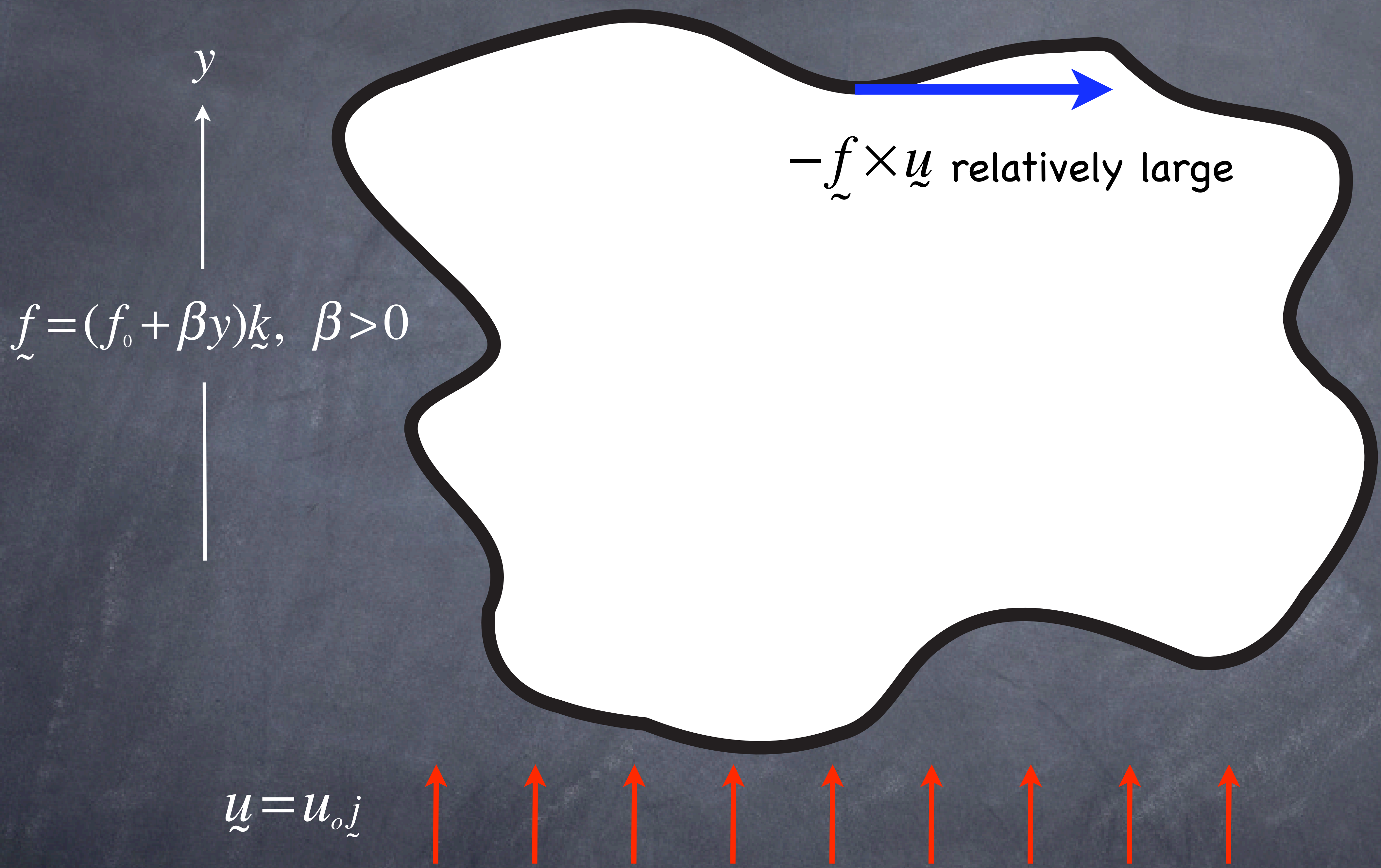

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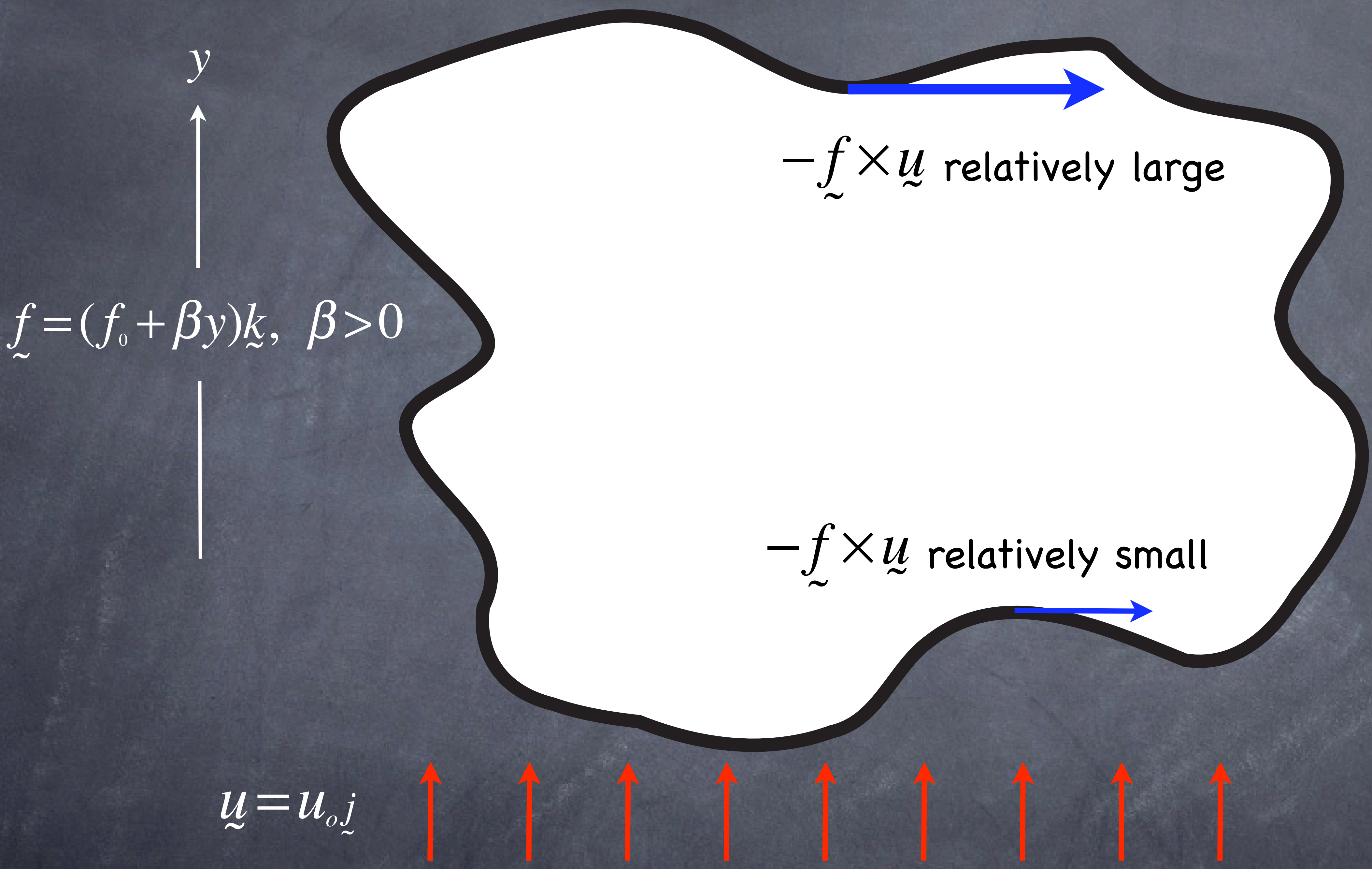

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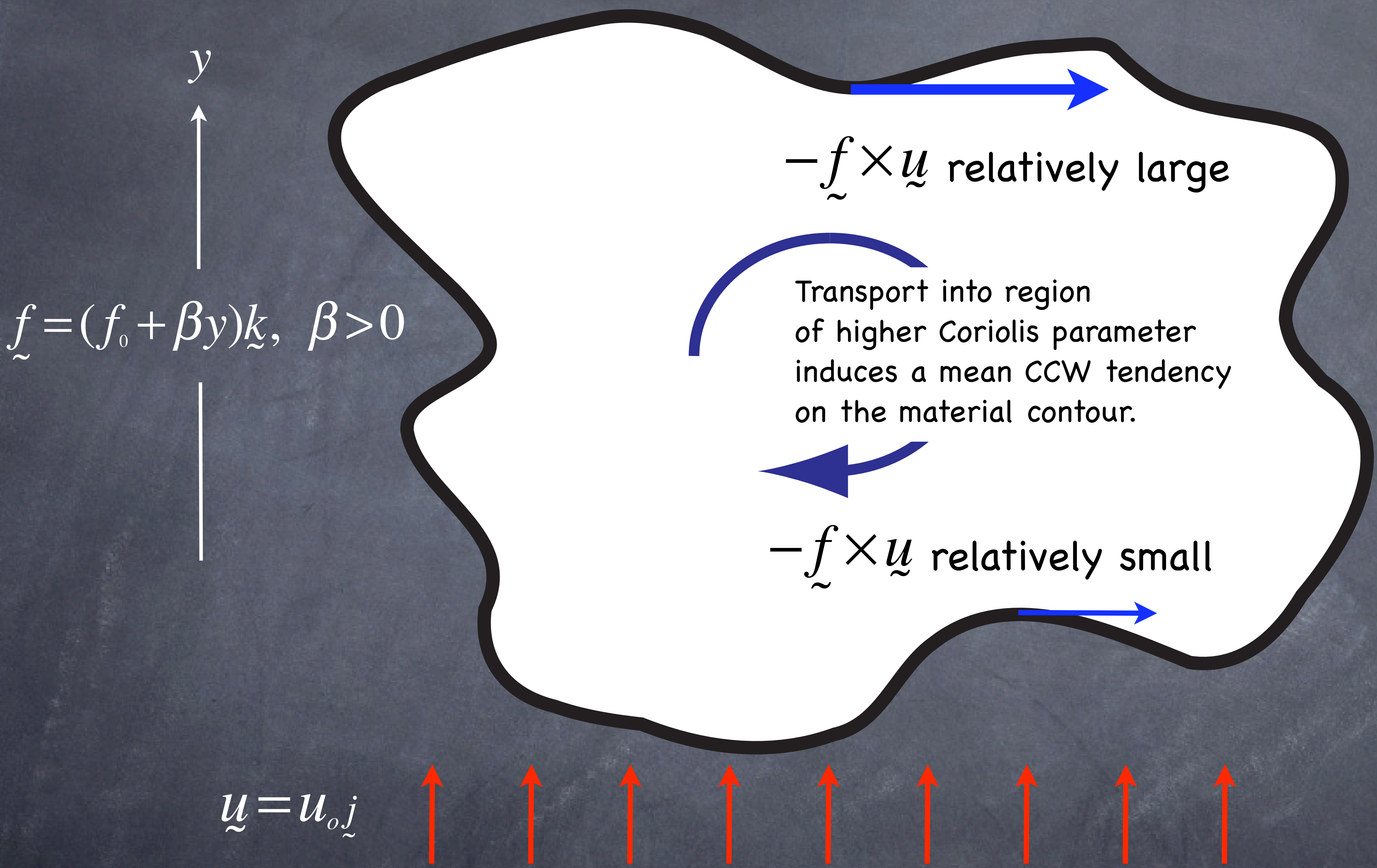

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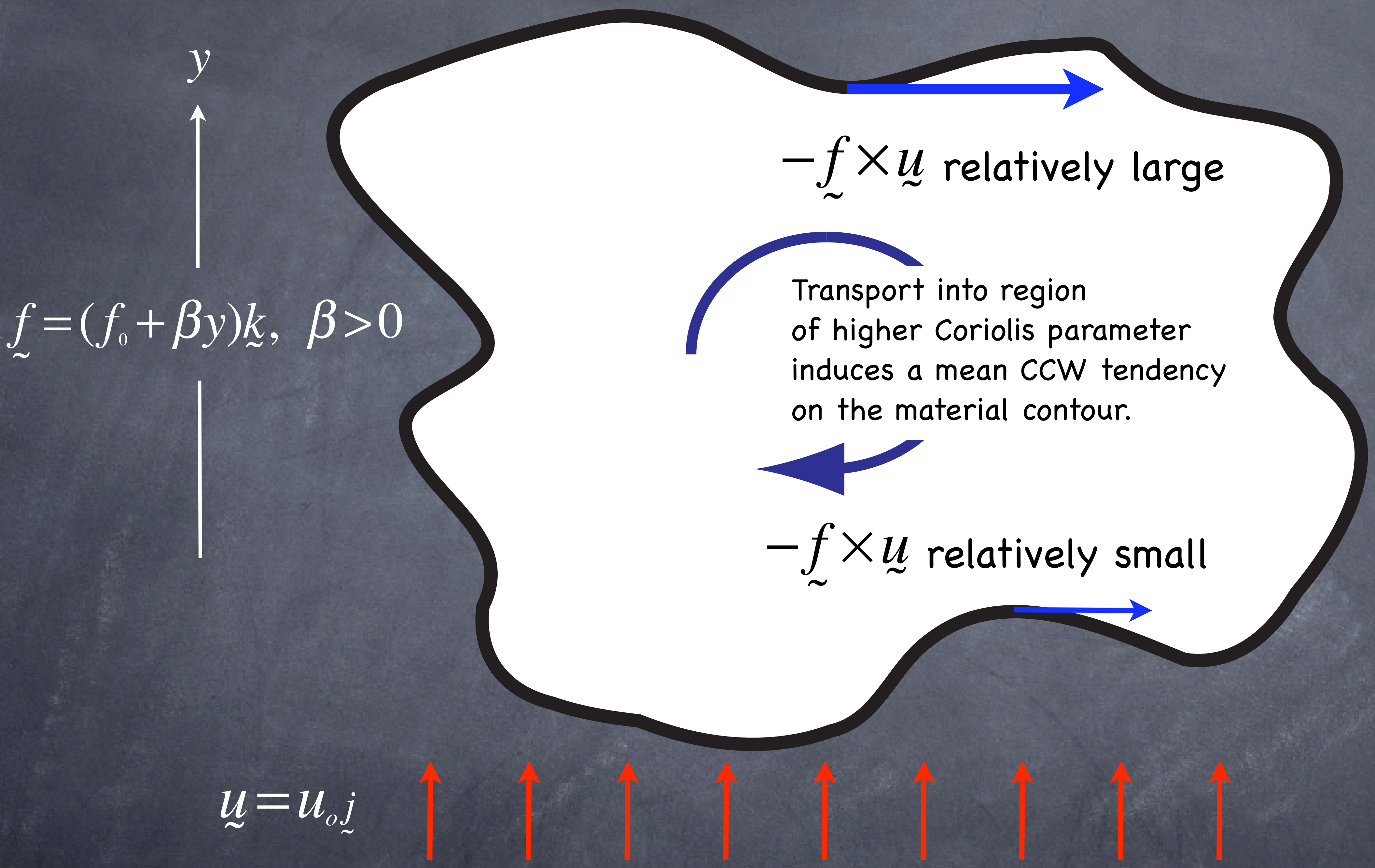
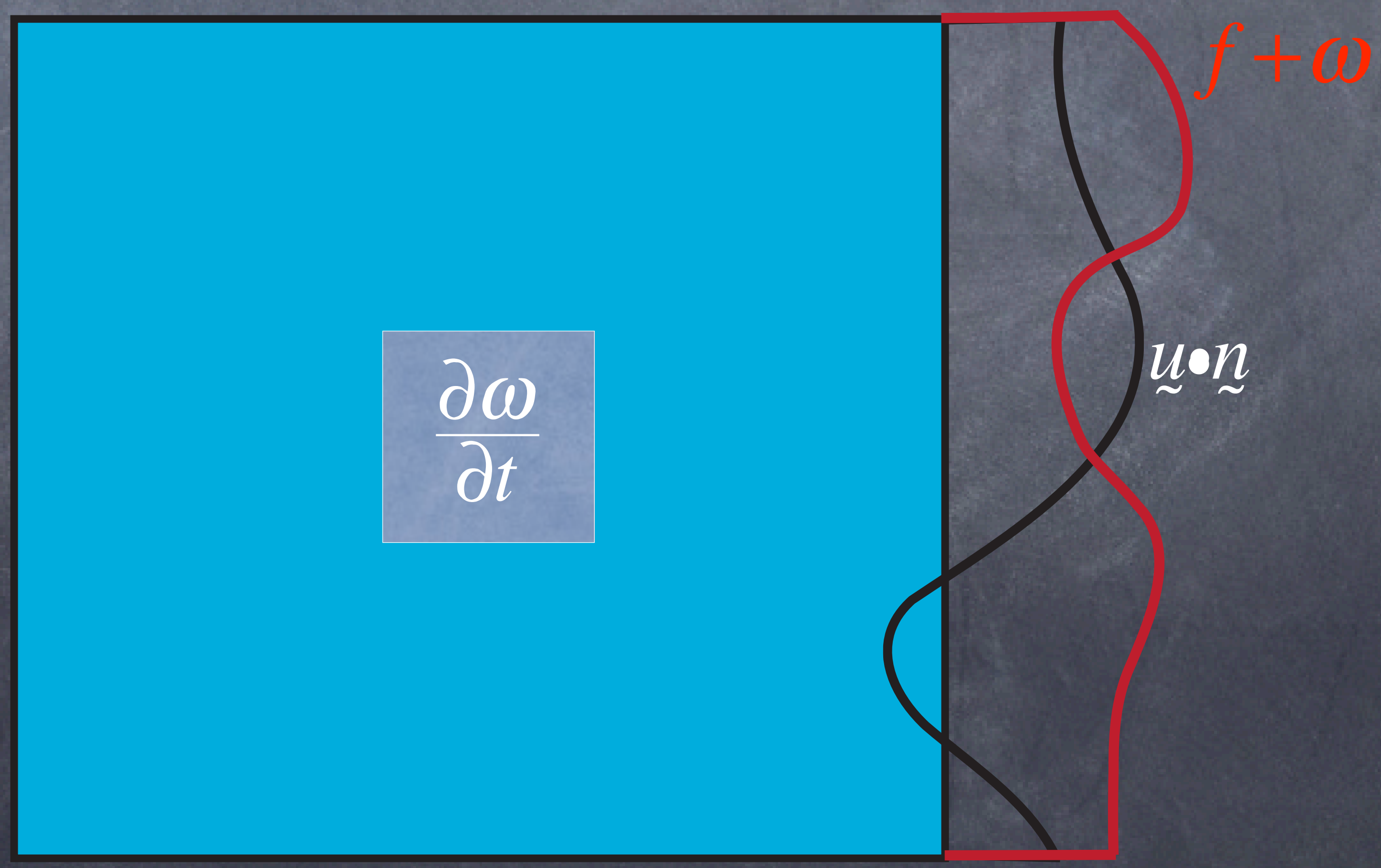


The tendency in circulation and mean tangential velocity is CCW.
The tendency in mean relative vorticity is negative.

Applying RTT to Circulation

$$\frac{D\Gamma_c}{Dt} = \frac{D}{Dt}\int_A \omega \, dA = \int_A \left[\frac{\partial \omega}{\partial t} + \nabla \bullet (\omega \underset{\sim}{u}) \right] dA = \oint_C -f\underset{\sim}{k} \times \underset{\sim}{u} \bullet d\underset{\sim}{s}$$



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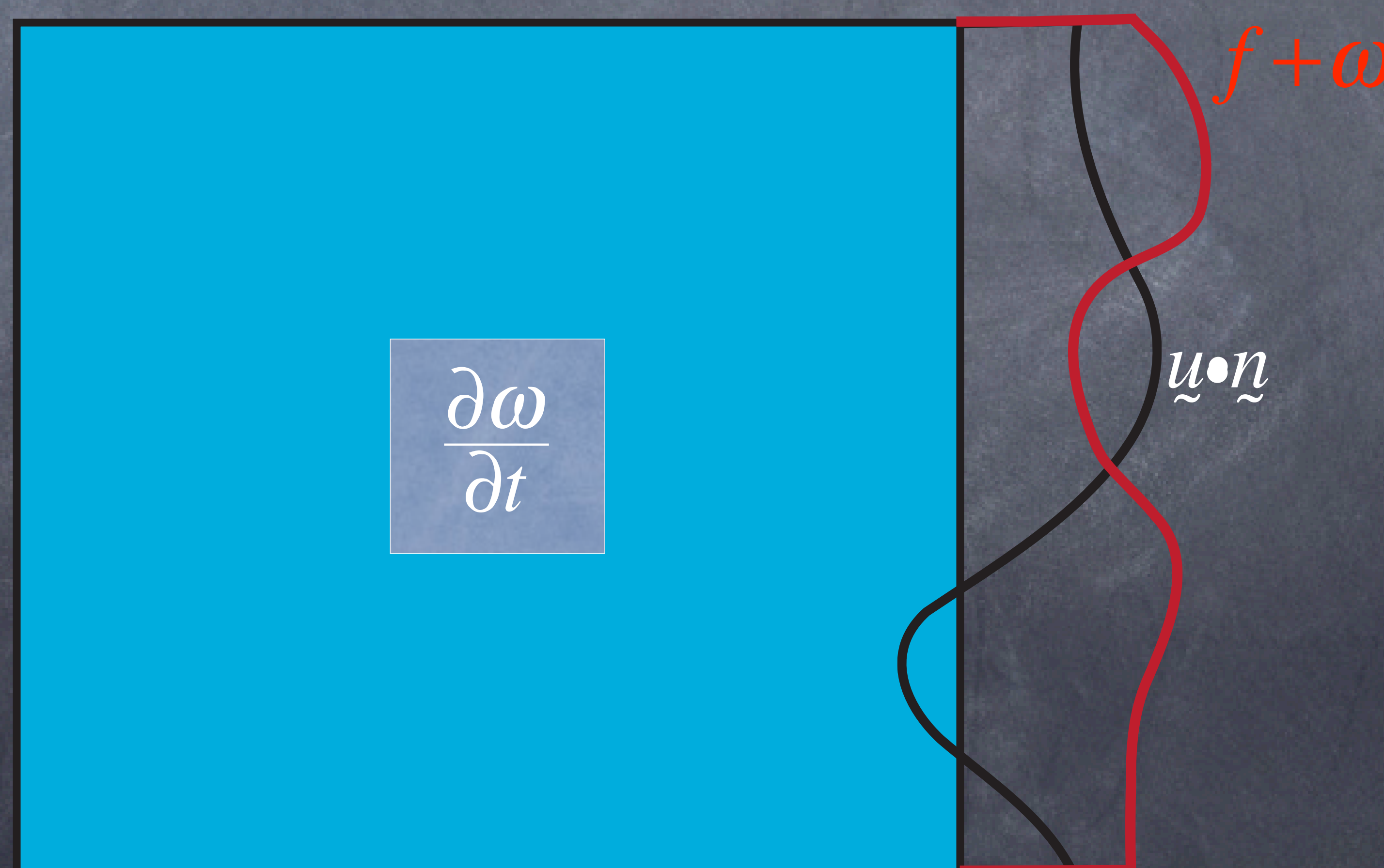

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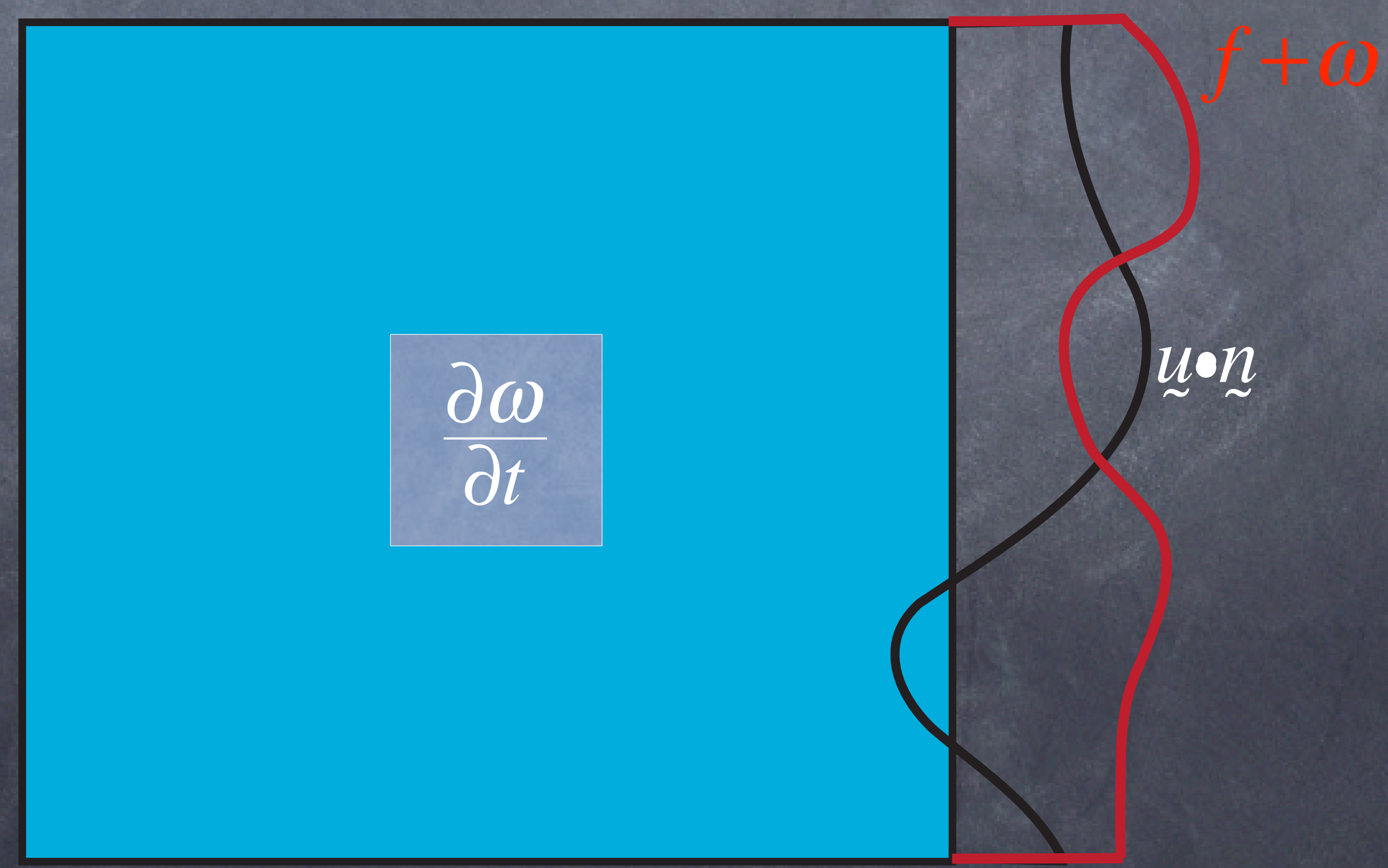
← Flux-form: What goes out of one cells goes into its neighbor.



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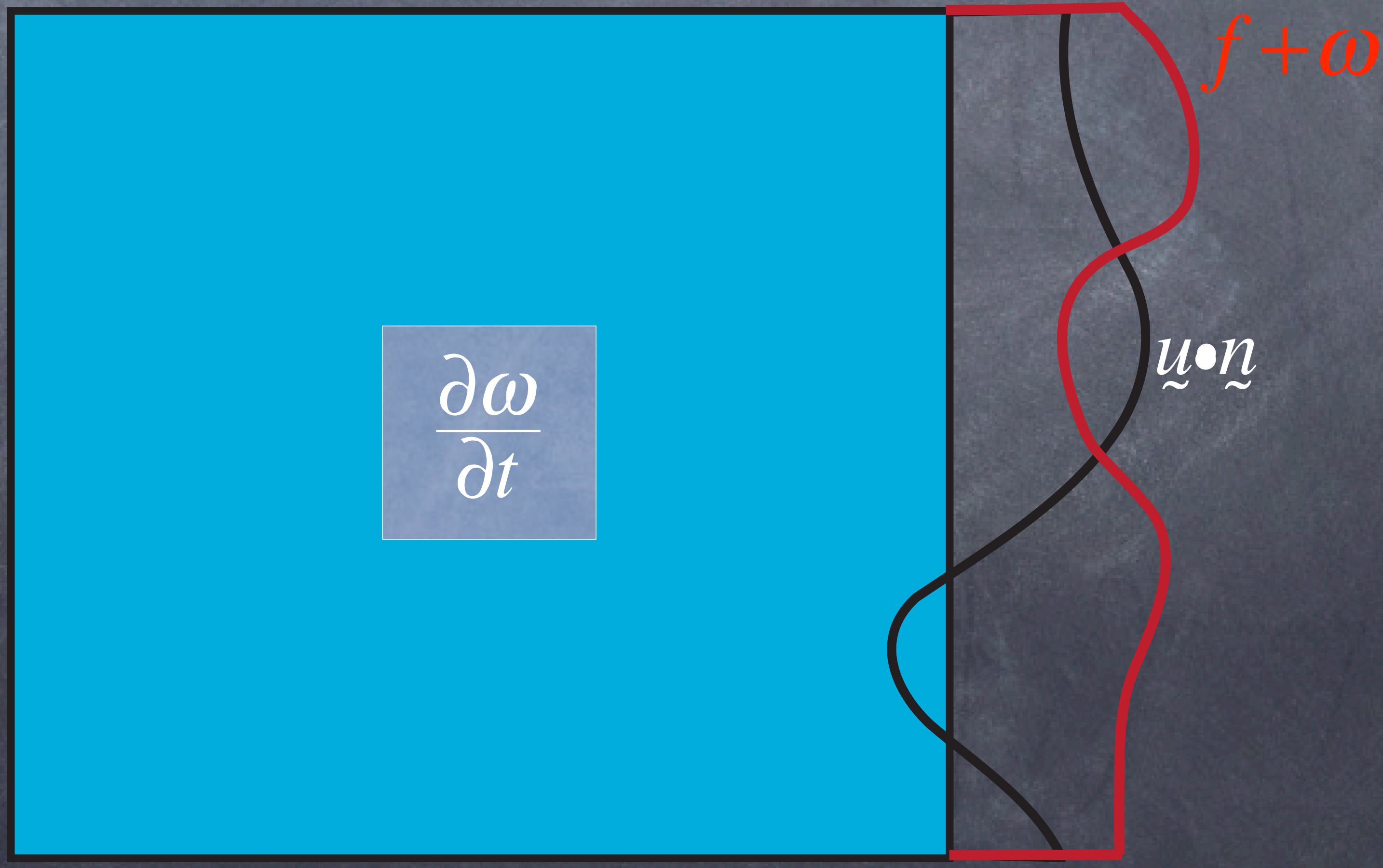
Flux-form: What goes out of one cells goes into its neighbor.

Within the control volume we can think of tracking the time tendency of any of the following: relative vorticity, absolute vorticity, circulation or even **contour-mean tangential velocity**.



Summary of Evolution Equations

$$\frac{D}{Dt}\left[\int_V \rho dV\right] = \int_V \frac{\partial \rho}{\partial t} dV + \int_S \rho\, \underset{\sim}{u} \bullet \underset{\sim}{n}\, dS = 0$$

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Each equation starts with a statement most naturally posed in a reference frame moving with the fluid, then using Reynold's Transport Theorem we recast that statement in a fixed reference frame conducive to present-day numerical methods.

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The momentum equation has various forms, each with its own advantages and disadvantages, let's take a closer look.

The various “flavors” of F=ma are all equivalent when expressed in their continuous form. The advantages and disadvantages are in the context of their discrete analogs.

The advective form of the momentum equation

$$\frac{D\underset{\sim}{u}}{Dt} = \frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \bullet \nabla \underset{\sim}{u} = -\underset{\sim}{f} \times \underset{\sim}{u} - \frac{1}{\rho} \nabla p + \nabla \bullet \underset{\approx}{T}$$

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Integrate along a particle path:

$$\underset{\sim}{u}(t_{end}) - \underset{\sim}{u}(t_{begin}) = dt \int_{t_{begin}}^{t_{end}} \left[-\underset{\sim}{f} \times \underset{\sim}{u} - \frac{1}{\rho}\nabla p + \nabla \bullet \underset{\approx}{T} \right]$$

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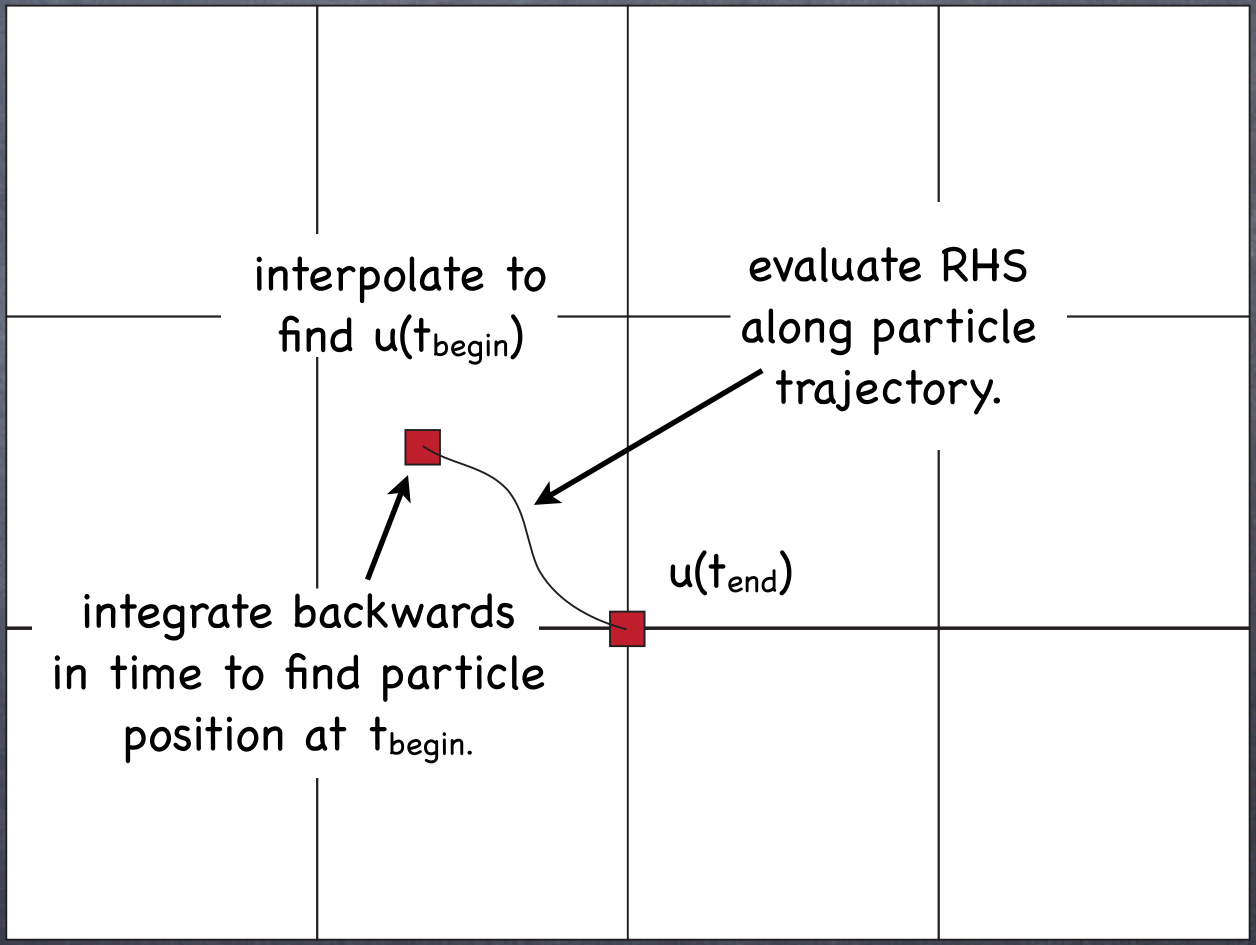

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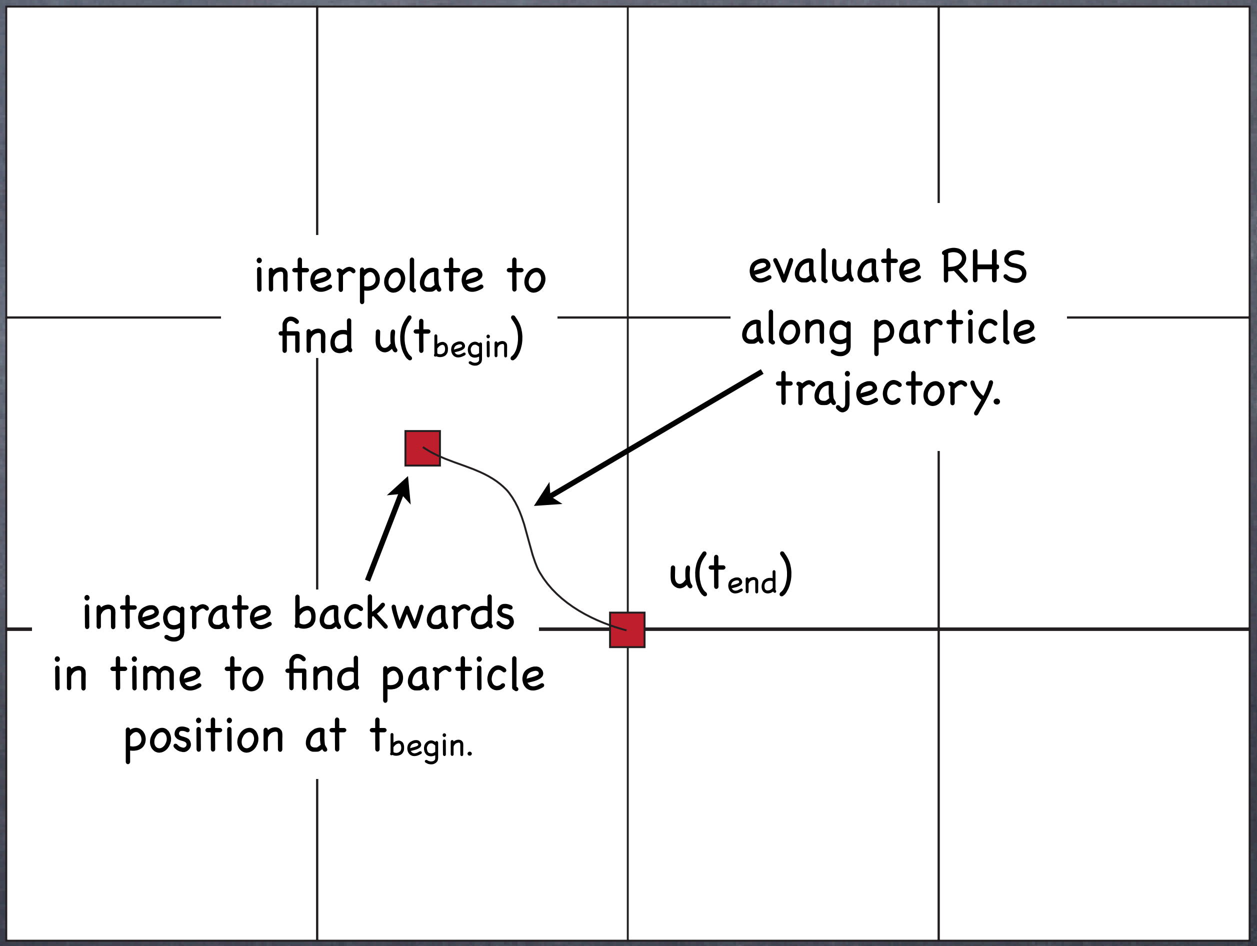
Advantages:

- Most natural expression of particle motion.

Disadvantages:

- Evaluation of integral along particle path is challenging.
- No solid handle on vorticity dynamics, including grad(p).



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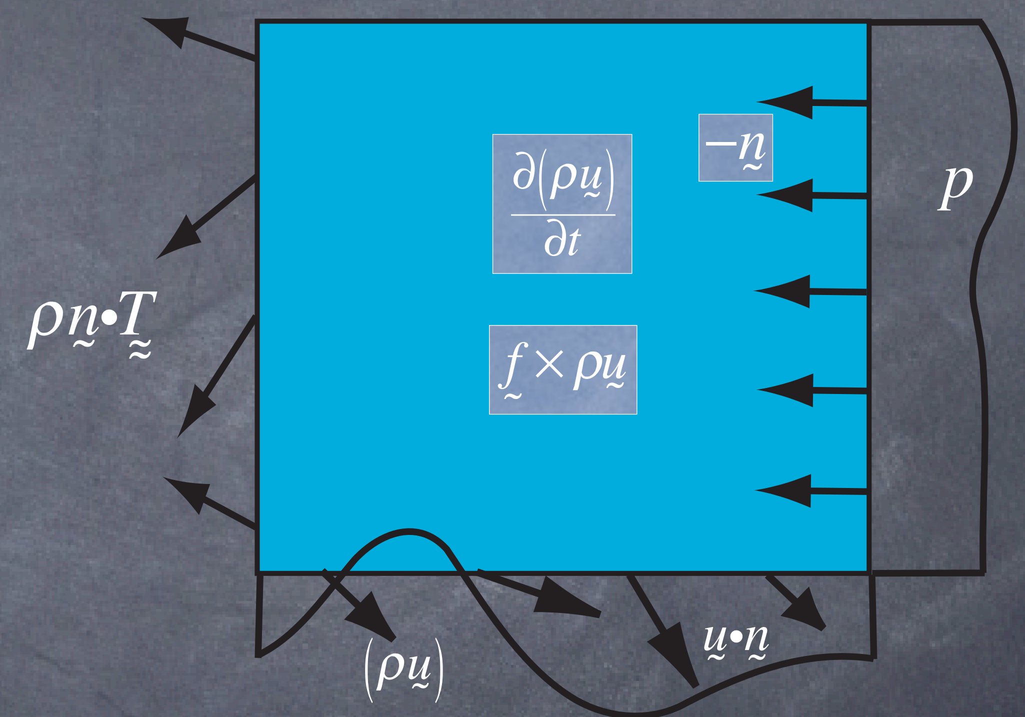

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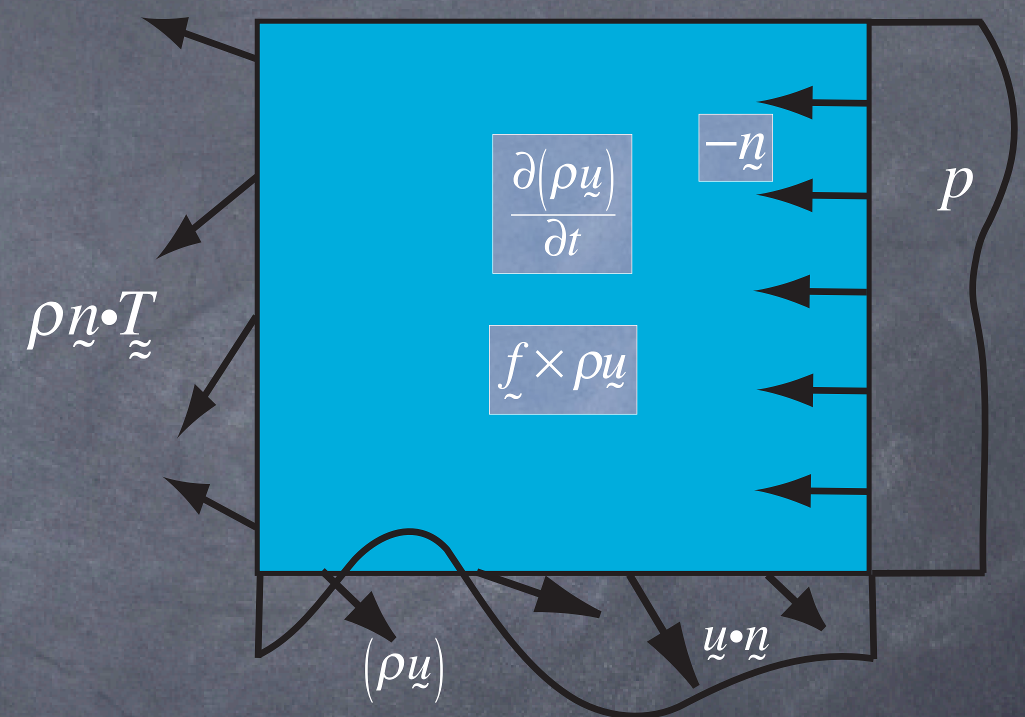
Advantages:

- Most natural expression of momentum equation for a control volume approach.
- Momentum equation is closely linked to mass equation (as it should be).
- As vertical layer thickness goes to zero, momentum naturally goes to zero.



Disadvantages:

- Line integral of tendency terms around control volume is not the circulation.
- Coriolis force is a volume integral whereas the pressure force is a surface integral.

The invariant form of the momentum equation

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \left(\underset{\sim}{\omega} + \underset{\sim}{f}\right) \times \underset{\sim}{u} = -\frac{1}{\rho}\nabla p - \frac{1}{2}\nabla \left\|\underset{\sim}{u}\right\|^2 + \nabla \bullet \underset{\approx}{T}$$

$$\left[\text{using } \underset{\sim}{u} \bullet \nabla \underset{\sim}{u} = \underset{\sim}{\omega} \times \underset{\sim}{u} + \frac{1}{2}\nabla \left\|\underset{\sim}{u}\right\|^2 \right]$$

Advantages:

- Most natural expression of momentum equation at a fixed-point (finite-difference).
- Equation contains the two primary derived quantities we are interested in (vorticity and kinetic energy).
- Line integral of tendency terms around a closed contour is the circulation.
- Rather difficult nonlinear advection term is recast as a cross-product and a gradient.

Disadvantages:

- It is my favorite form, so there are no disadvantages!

The vor/div form of the momentum equation

If vorticity is what we are interested in, why not just exchange momentum for its vorticity and divergence?

•

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Advantages:

Direct handle on the evolution of vorticity and divergence.

Disadvantages:

1st order spatial derivatives are now 2nd order derivatives.

Additional boundary conditions required to constrain higher derivs

Invert elliptic equations to determine velocity.

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An analysis of F=ma in a discrete setting.

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We have (at least):

- 4 forms of F=ma: advect, flux, invariant, vor/div
- 3 classes of meshes: quads, triangles, hexagons.
- 6 methods (just in finite volume): A, B, C, D, E, Z

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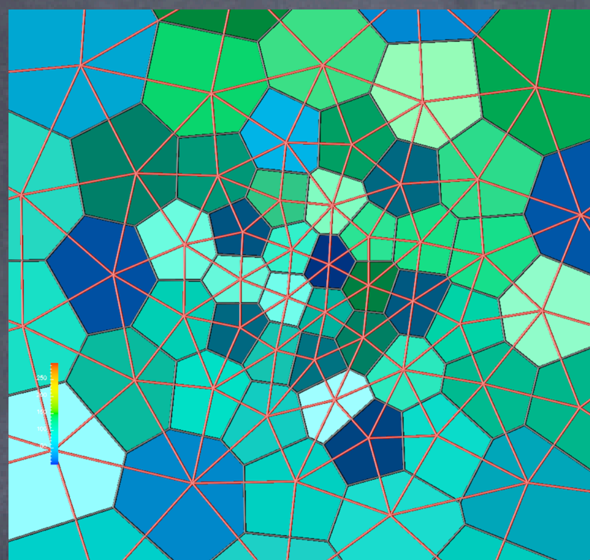
Why? This combination provides precise control of vortex dynamics and energetics, results in acceptable gravity wave simulation and is applicable to any locally-orthogonal mesh (either quasi-uniform or variable resolution).

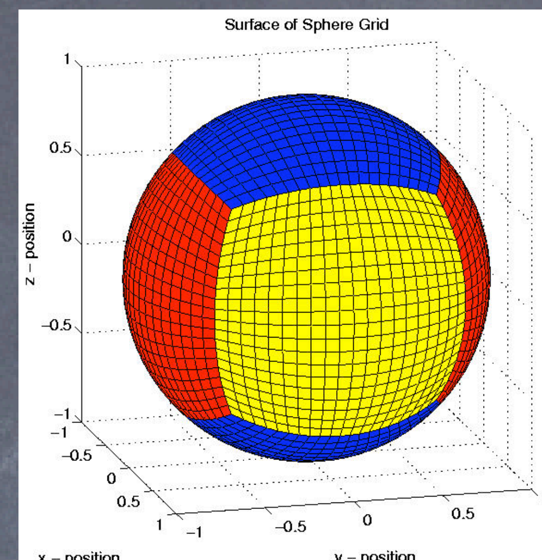
What does it mean to be locally-orthogonal?

The dot product of vectors tangent to relevant lines segments is zero.

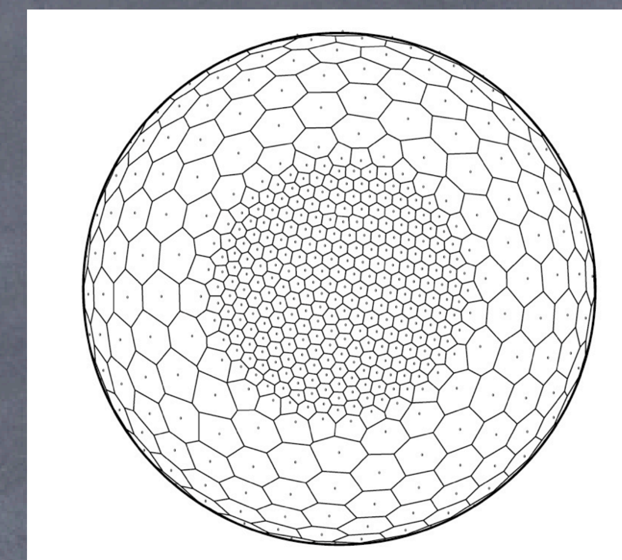
This captures a wide cross-section of meshes.



Delaunay Triangulation

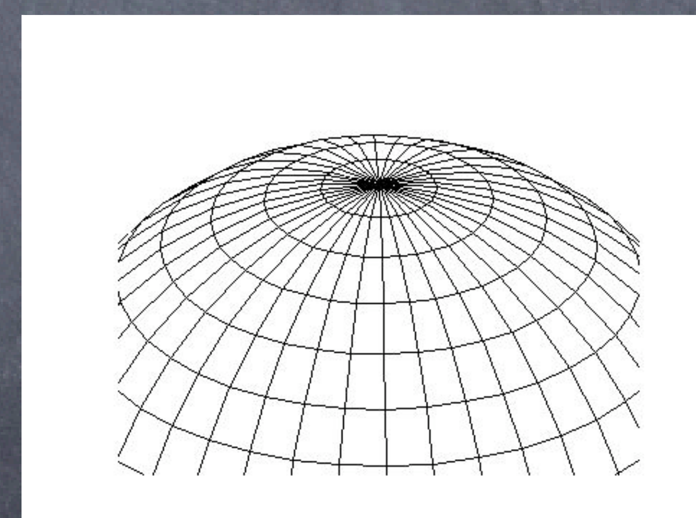


Cubed Sphere



Voronoi Diagram

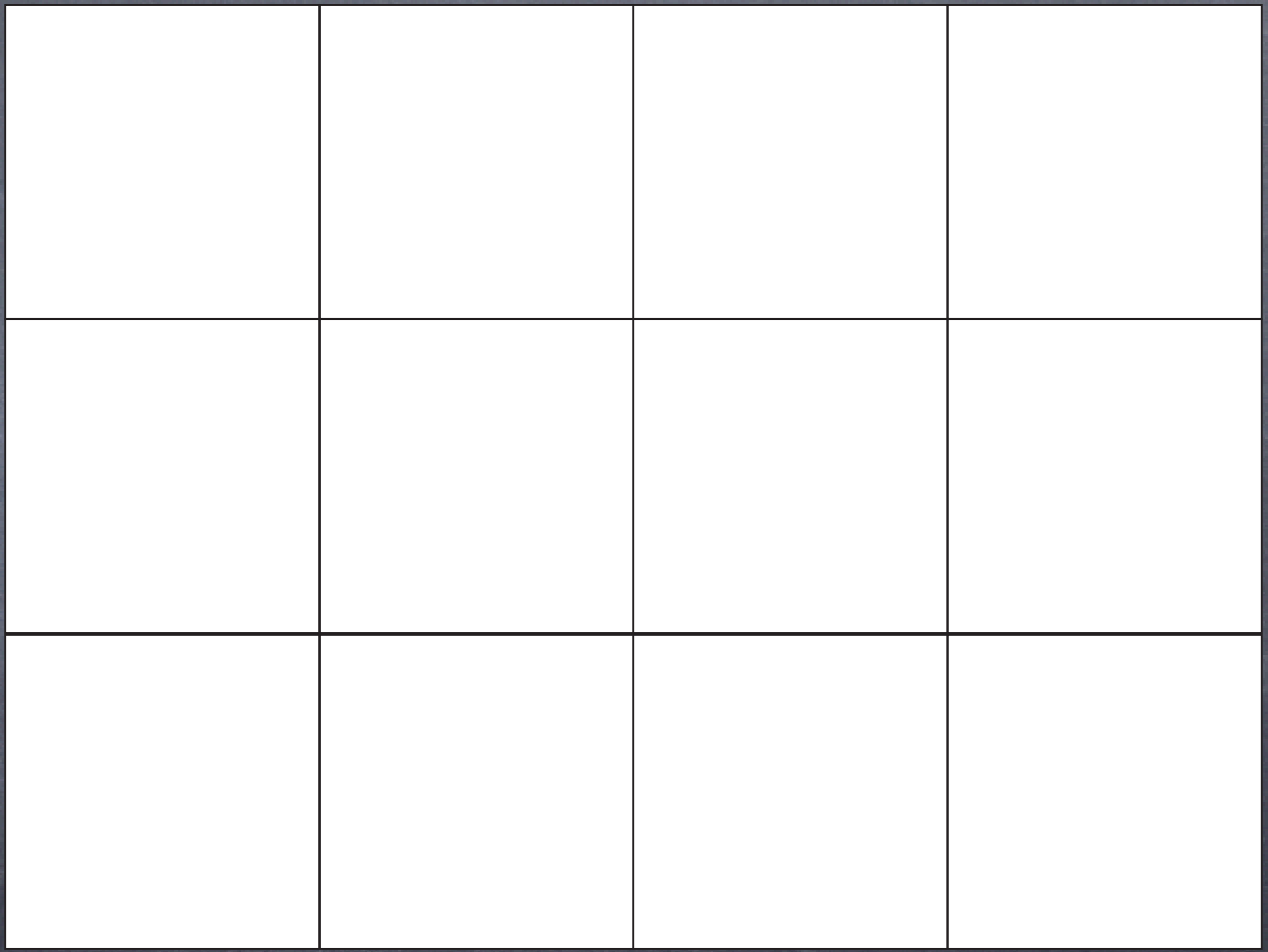
Stretched, Tri-pole grid



Latitude-Longitude

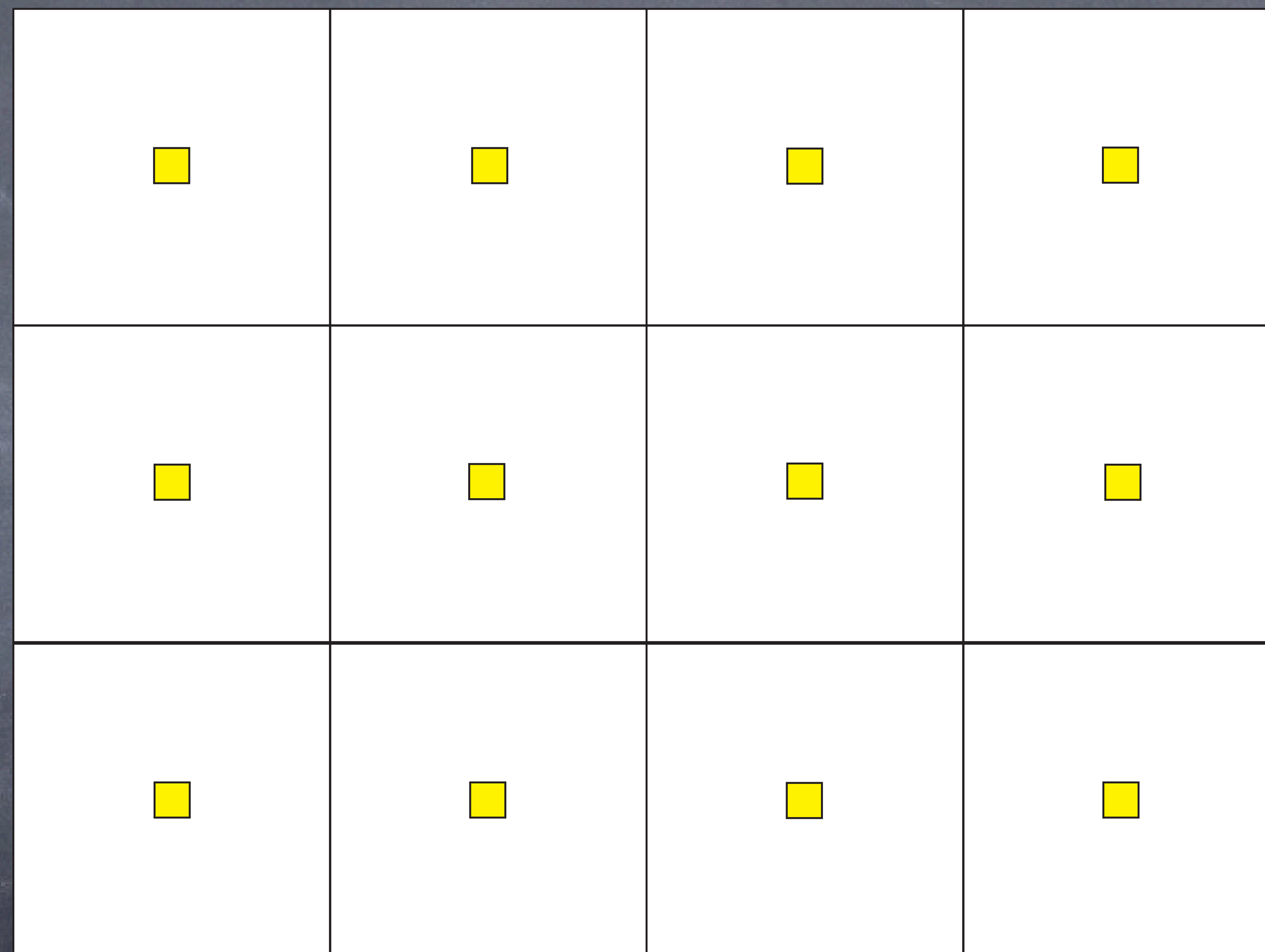
What follows is applicable of any of these meshes.

What is the C-grid staggering?



What is the C-grid staggering?

mass point



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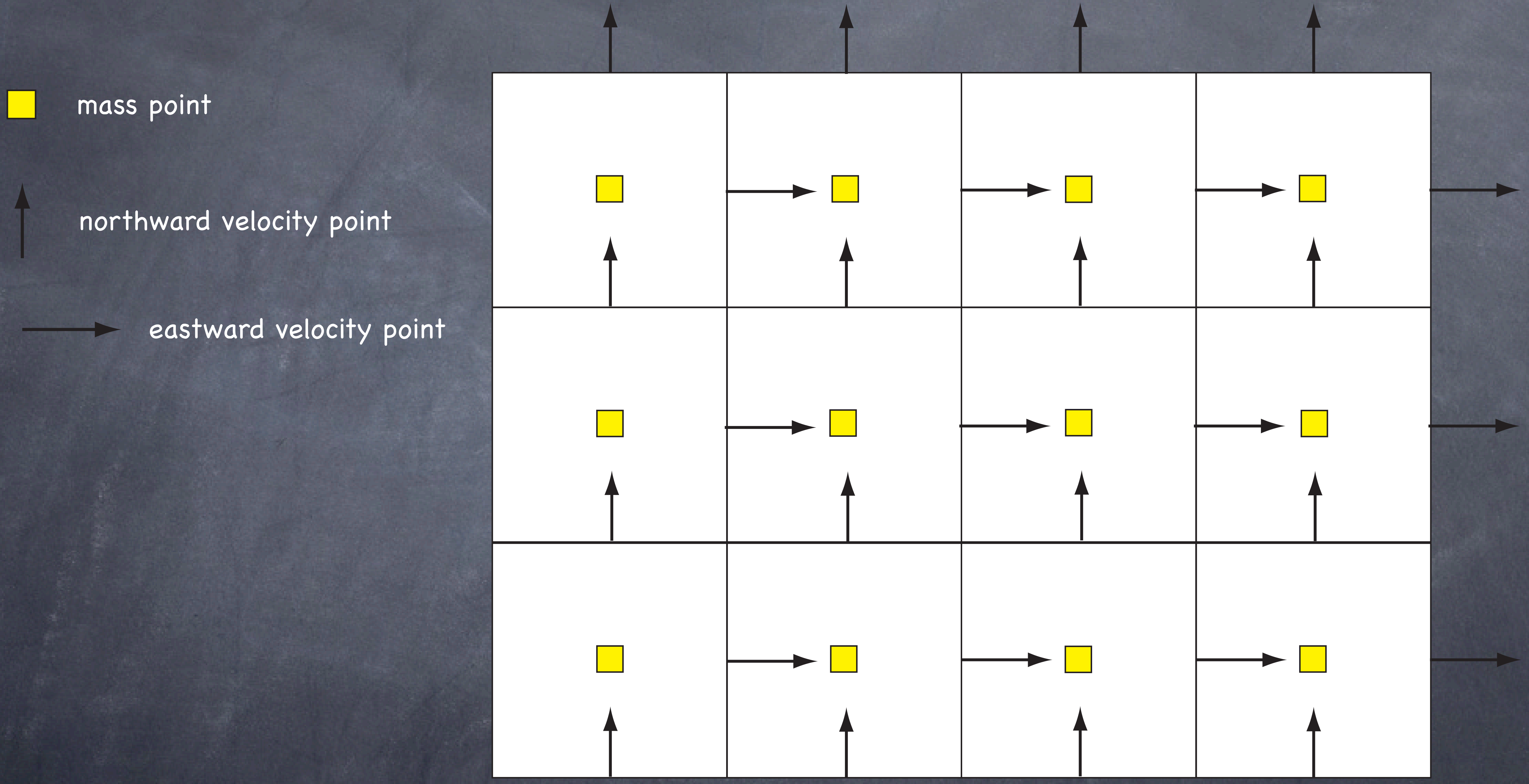

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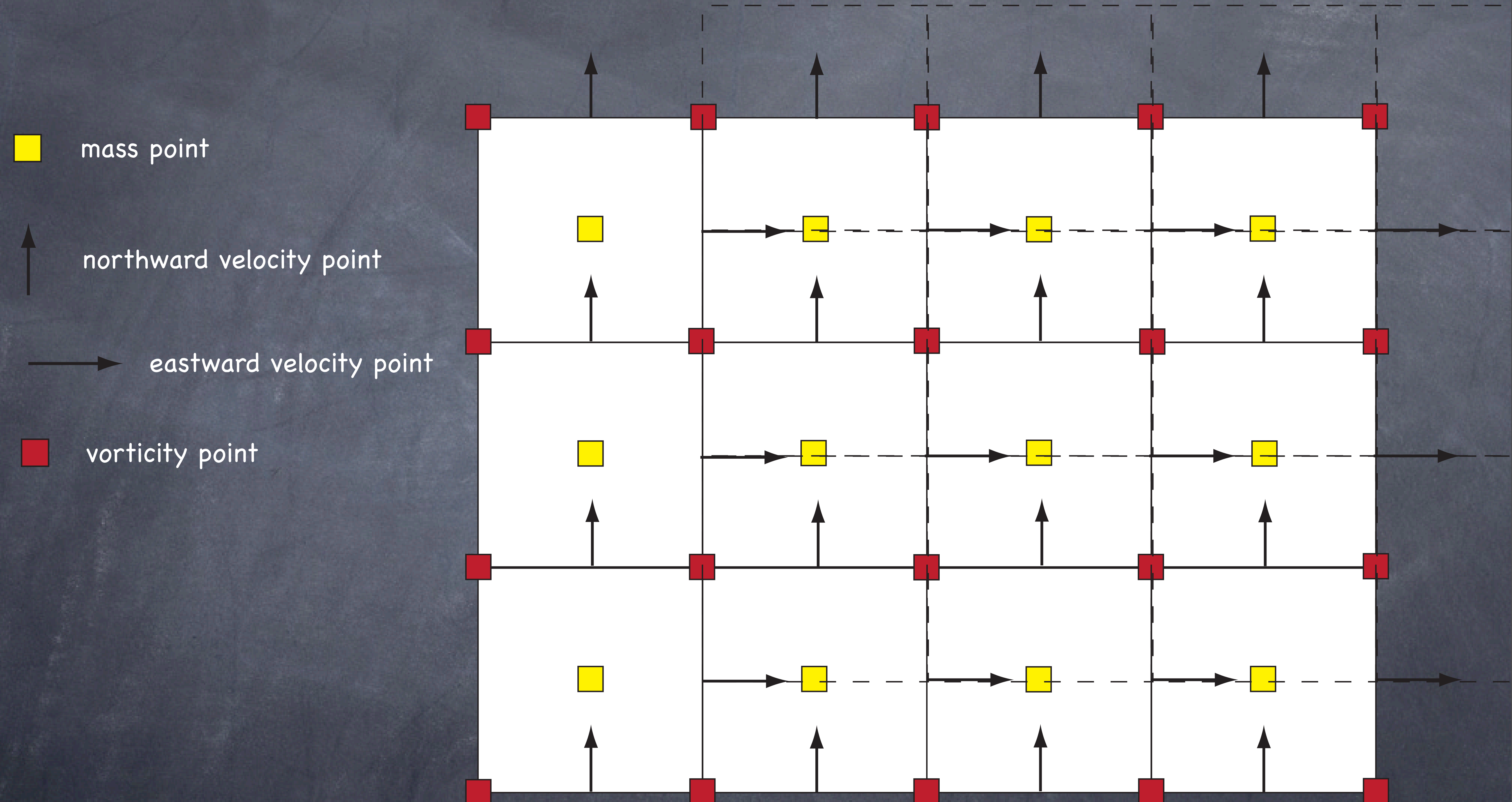

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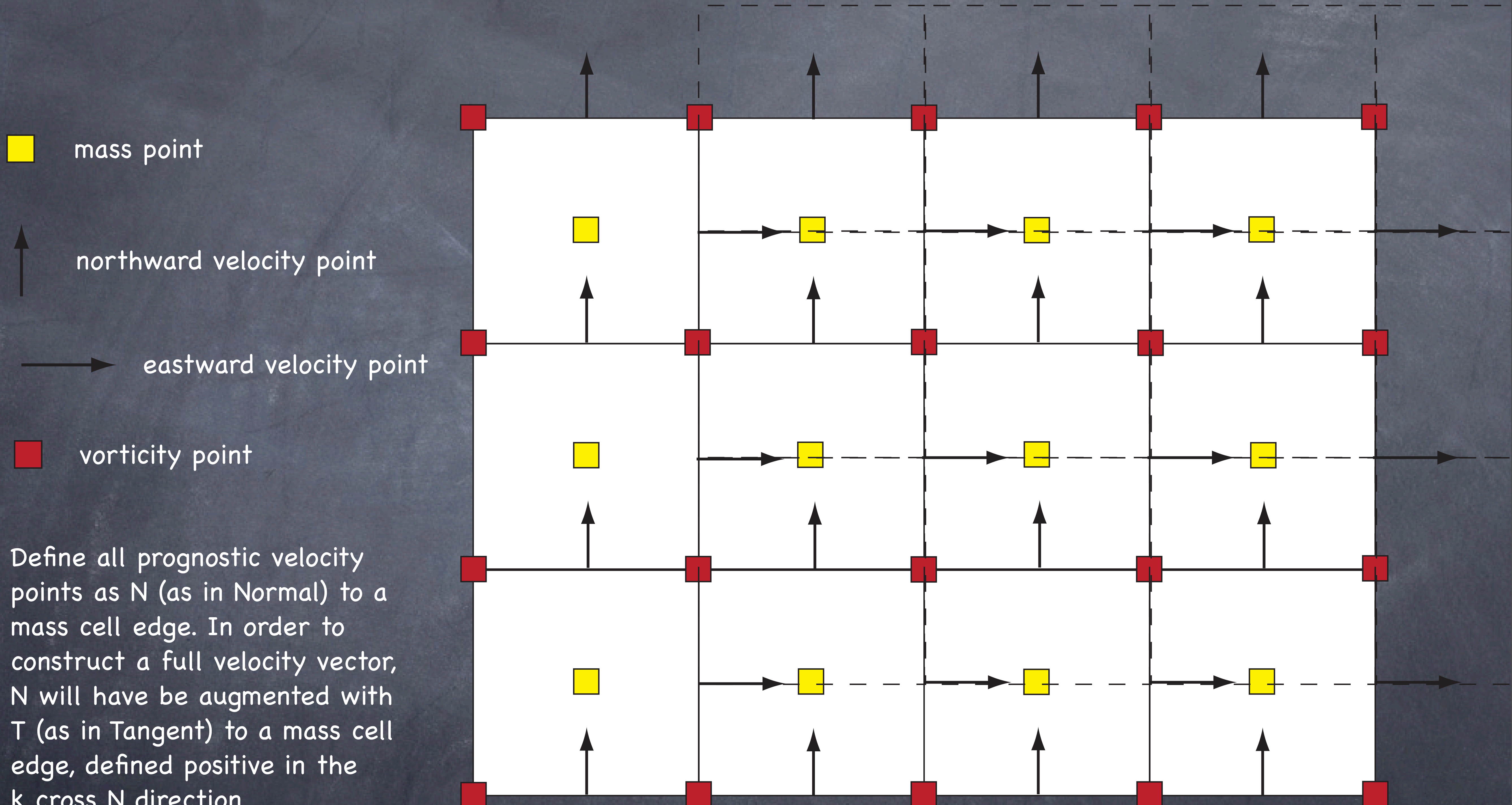

Define all prognostic velocity points as N (as in Normal) to a mass cell edge. In order to construct a full velocity vector, N will have be augmented with T (as in Tangent) to a mass cell edge, defined positive in the k cross N direction.

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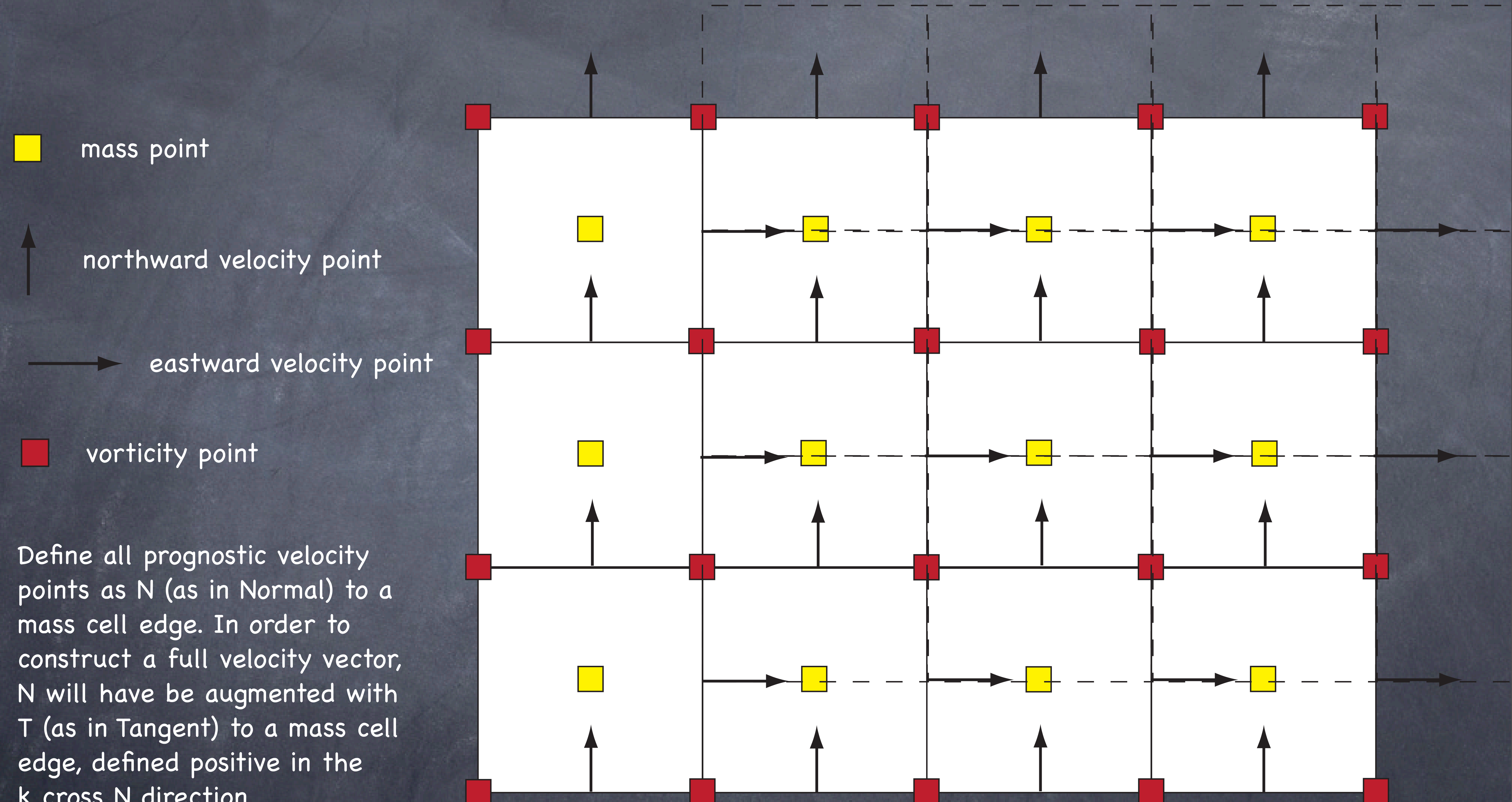

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The orthogonality constraint requires the line connecting two mass points to be orthogonal to the shared edge (and thus parallel to the projected velocity component.

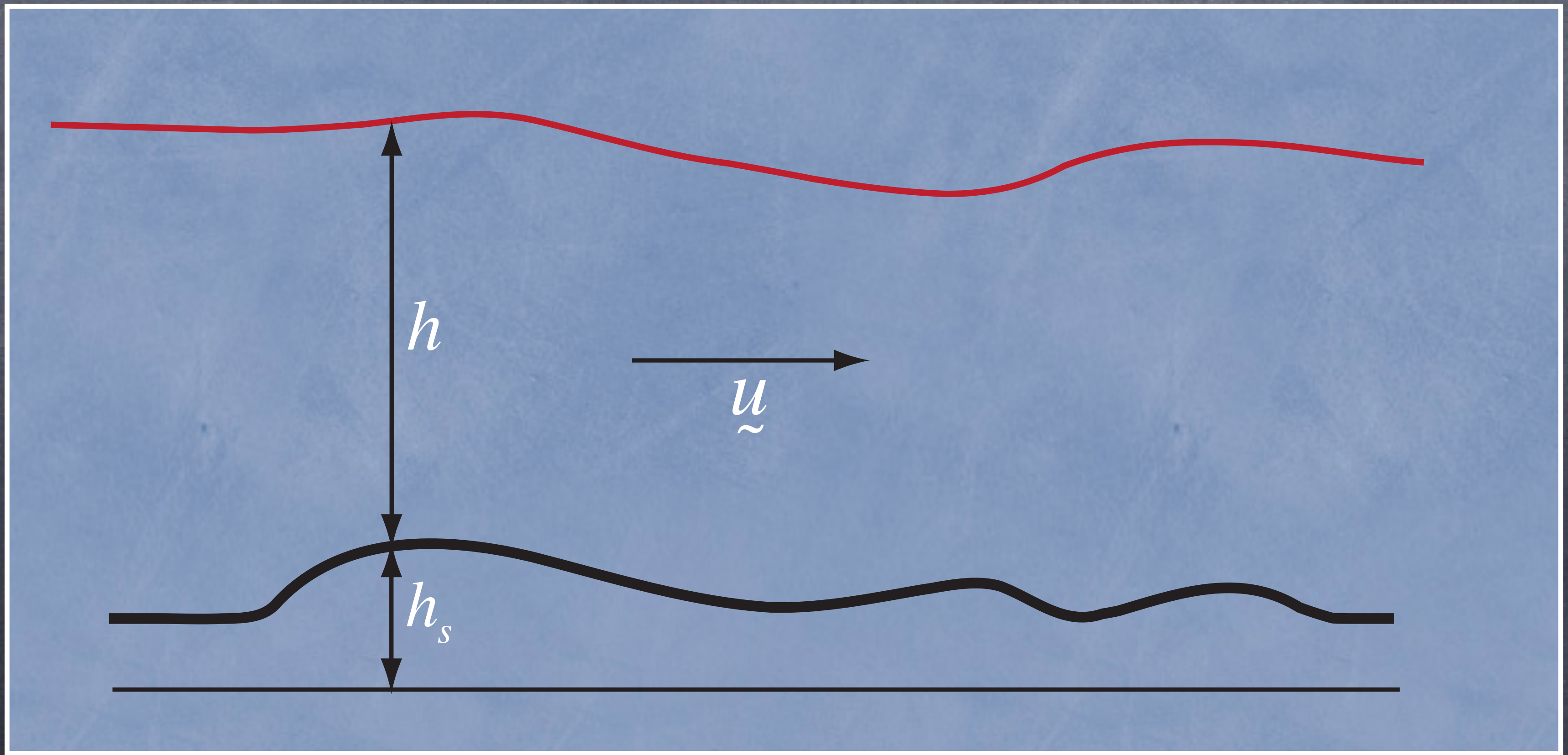
For clarity, let's simplify to the shallow-water equations.

$$\frac{\partial h}{\partial t} + \nabla \bullet (h\underset{\sim}{u}) = 0$$ thickness plays the role of pressure (uniform density).

$$\frac{\partial \underset{\sim}{u}}{\partial t} + (\omega + f)\underset{\sim}{k} \times \underset{\sim}{u} = -g\nabla(h + h_s) - \frac{1}{2}\nabla \|\underset{\sim}{u}\|^2$$ velocity is in the tangent plane.

$$\omega = \underset{\sim}{k} \bullet (\nabla \times \underset{\sim}{u})$$ only keep track of the vertical component of vorticity.

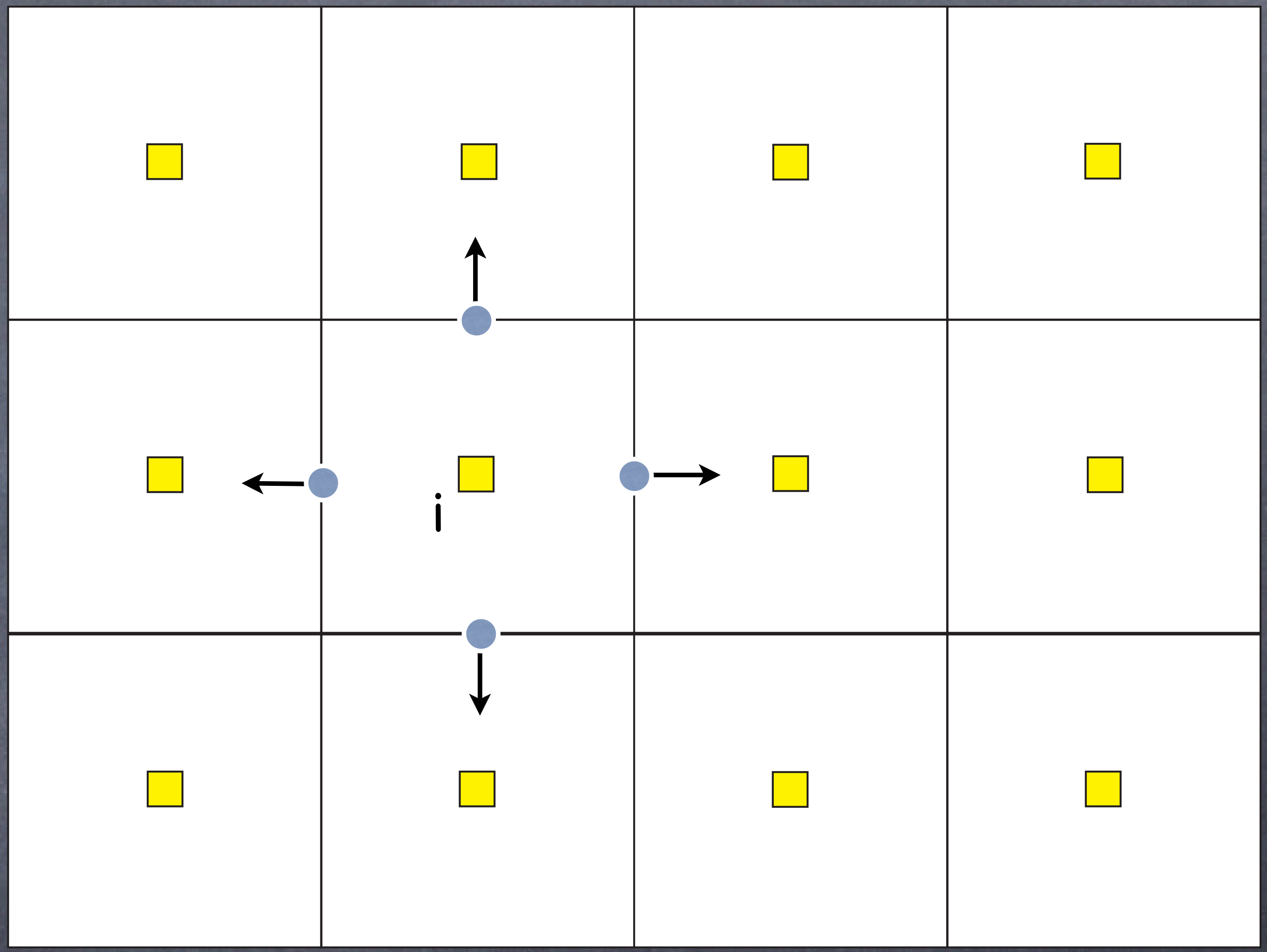
$$\eta = \omega + f$$ definition of absolute vorticity.



The discrete mass equation:
everything with an overhat has to be defined

$$\frac{\partial h_i}{\partial t} = \frac{-1}{A_i} \sum_{j=1}^{nedges} \hat{h}_j N_j \, dl_j$$

- A_i mass cell area
- dl_j cell edge length
- N_j normal velocity points out of cell i
- $\hat{h}_j$ thickness at cell edge

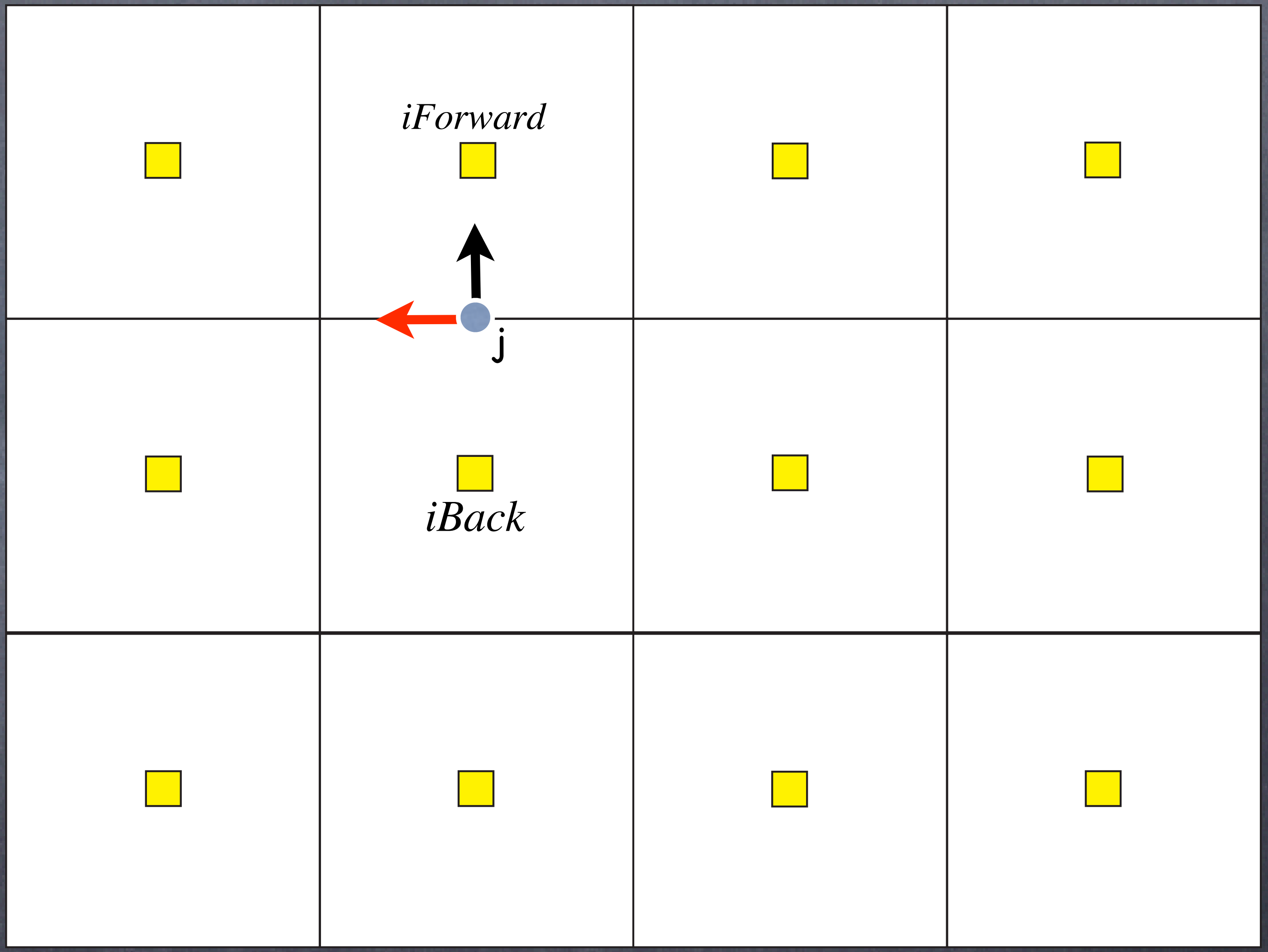


mass point

The discrete momentum equation: everything with an overhat has to be defined

$$\frac{\partial N_j}{\partial t} = \hat{\eta}_j \hat{T}_j - \left\{ \left[gh + gh_s + \hat{K} \right]_{iForward} - \left[gh + gh_s + \hat{K} \right]_{iBack} \right\} / dc_j$$

- dc_j distance between iForward and iBack
- $\hat{\eta}_j$ absolute vorticity
- $\hat{T}_j$ reconstructed, tangent velocity, for here simply state $\hat{T}_j = f(N_j)$.
- $gh + gh_s + \hat{K}$ sum of potential and kinetic energy



mass point

The discrete vorticity equation:

taking the curl of the momentum equation.

$$\frac{1}{A_k}\sum_{j=1}^{nedges} dc_j \left\{ \frac{\partial N_j}{\partial t} = \hat{\eta}_j \hat{T}_j - \left\{ \left[gh + gh_s + \hat{K} \right]_{iForward} - \left[gh + gh_s + \hat{K} \right]_{iBack} \right\} / dc_j \right\}$$

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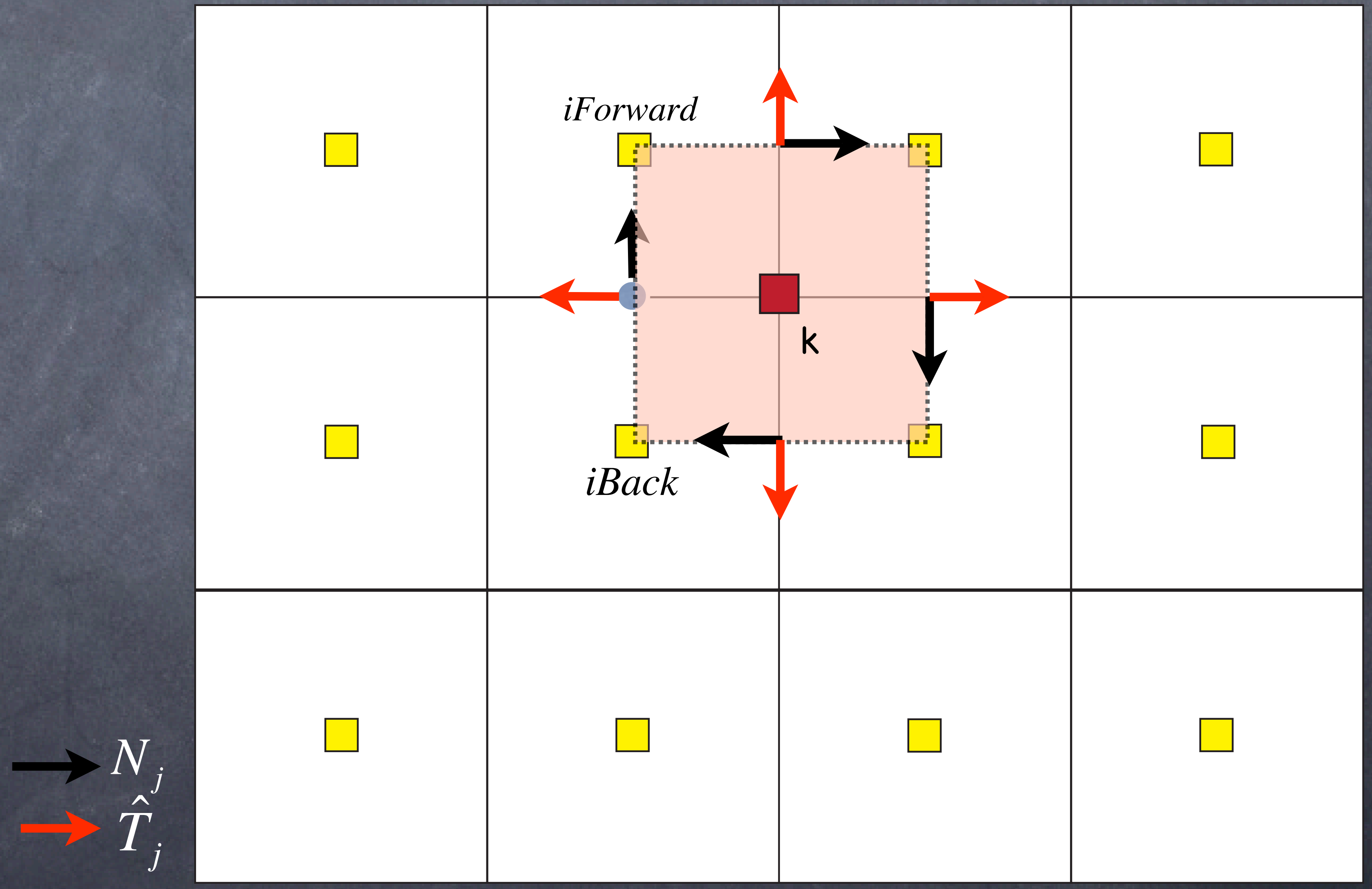

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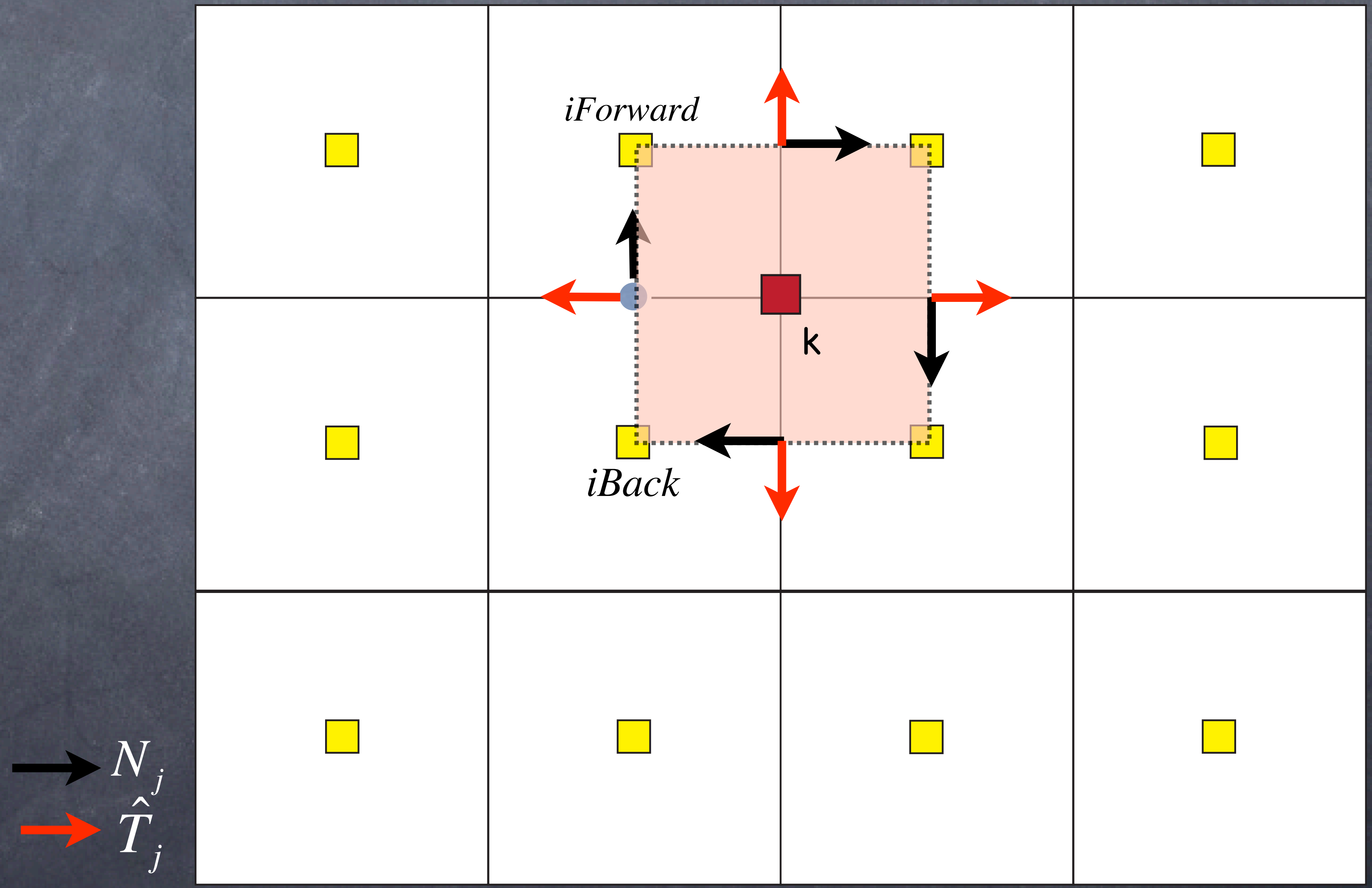

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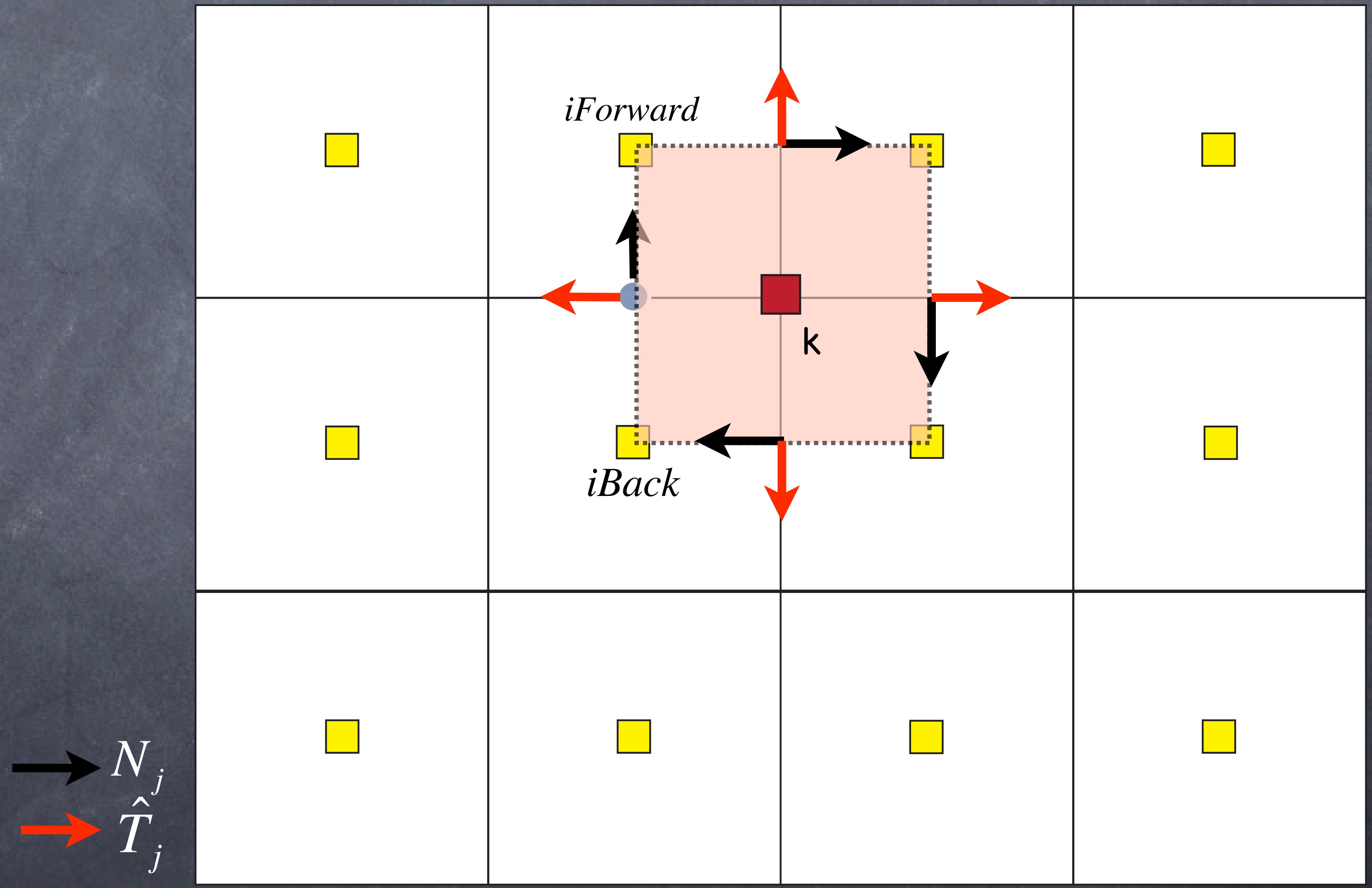

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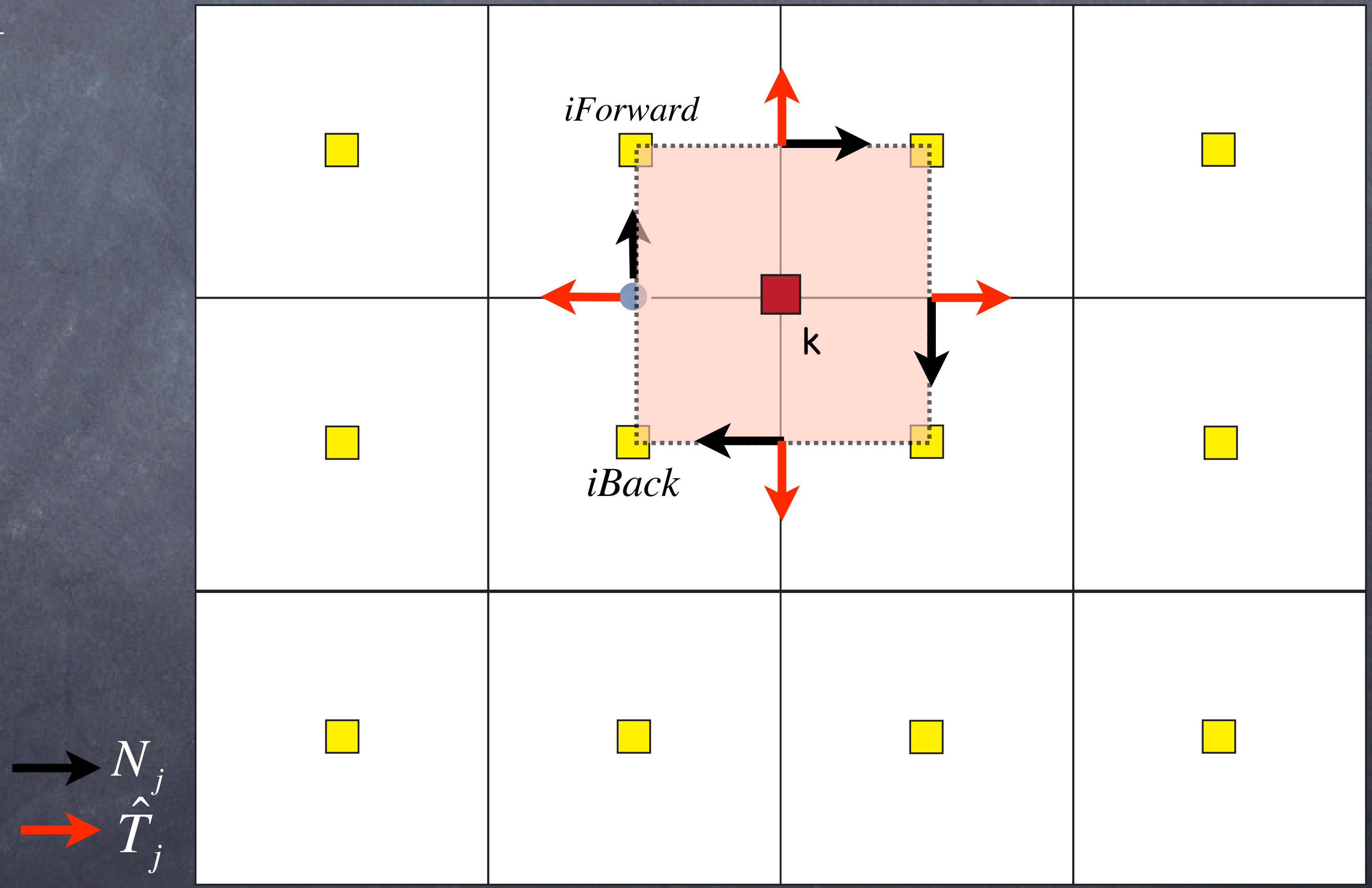

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Our discrete momentum equation **dictates** the following discrete vorticity equation:

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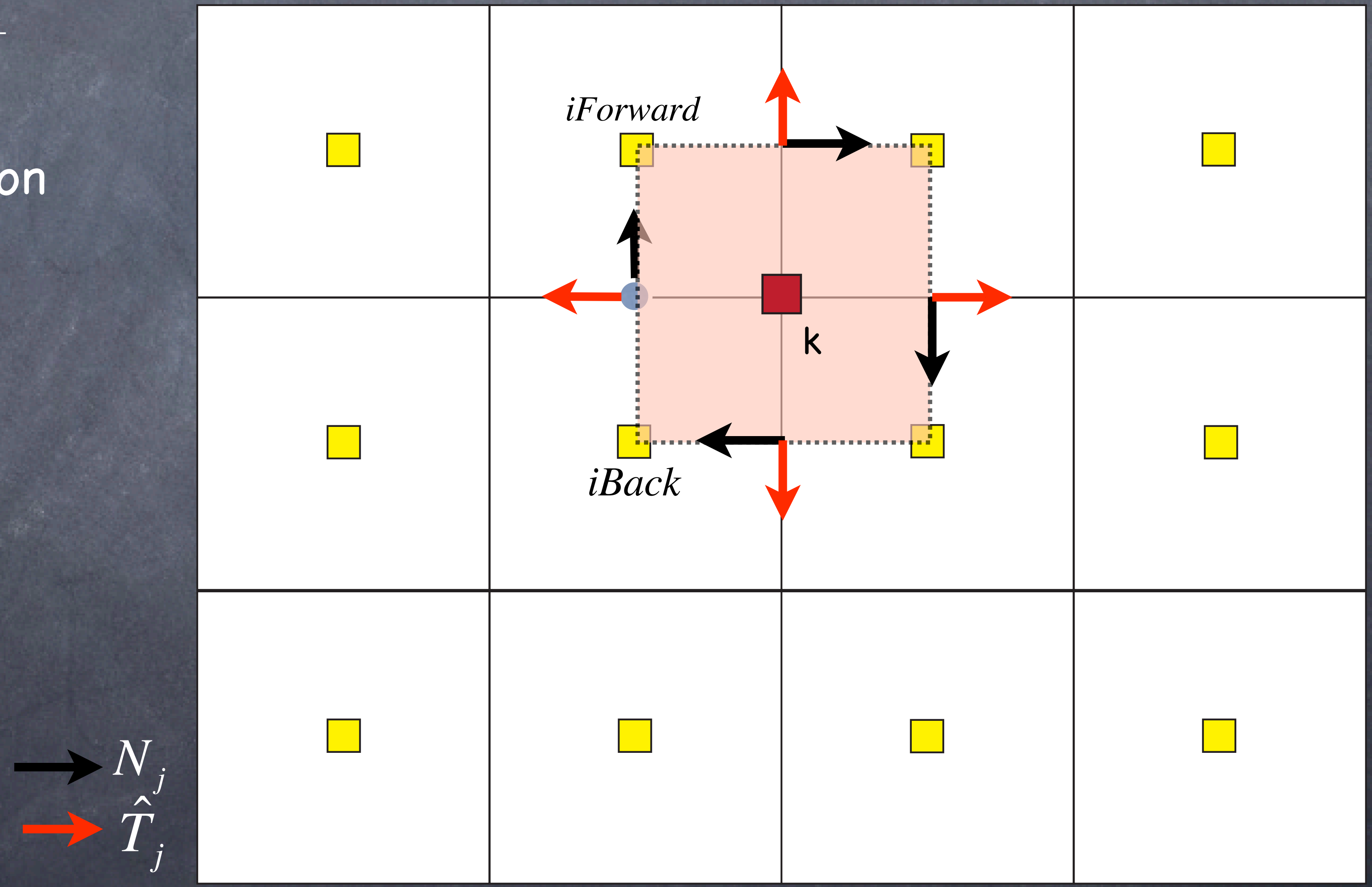

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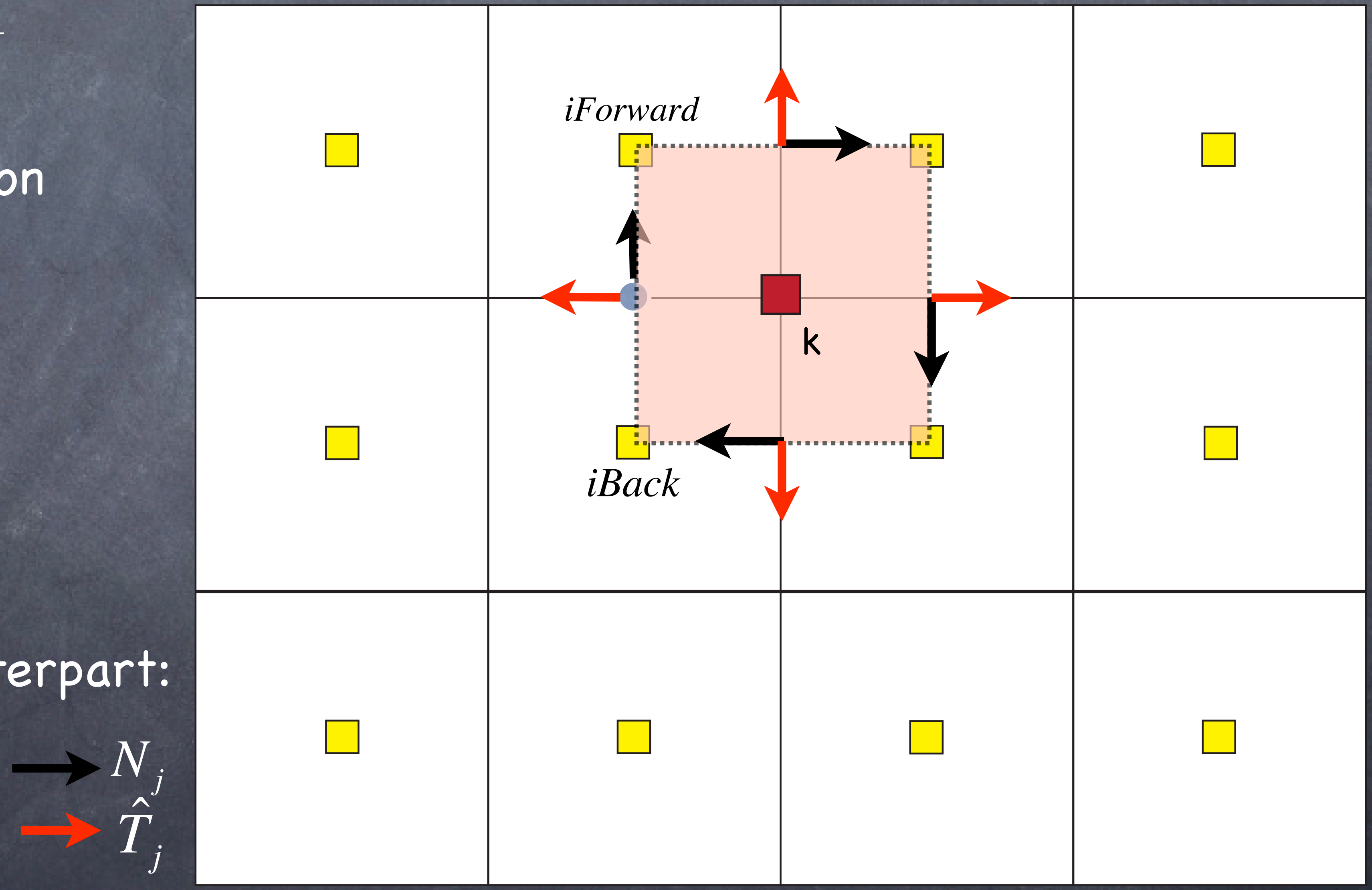
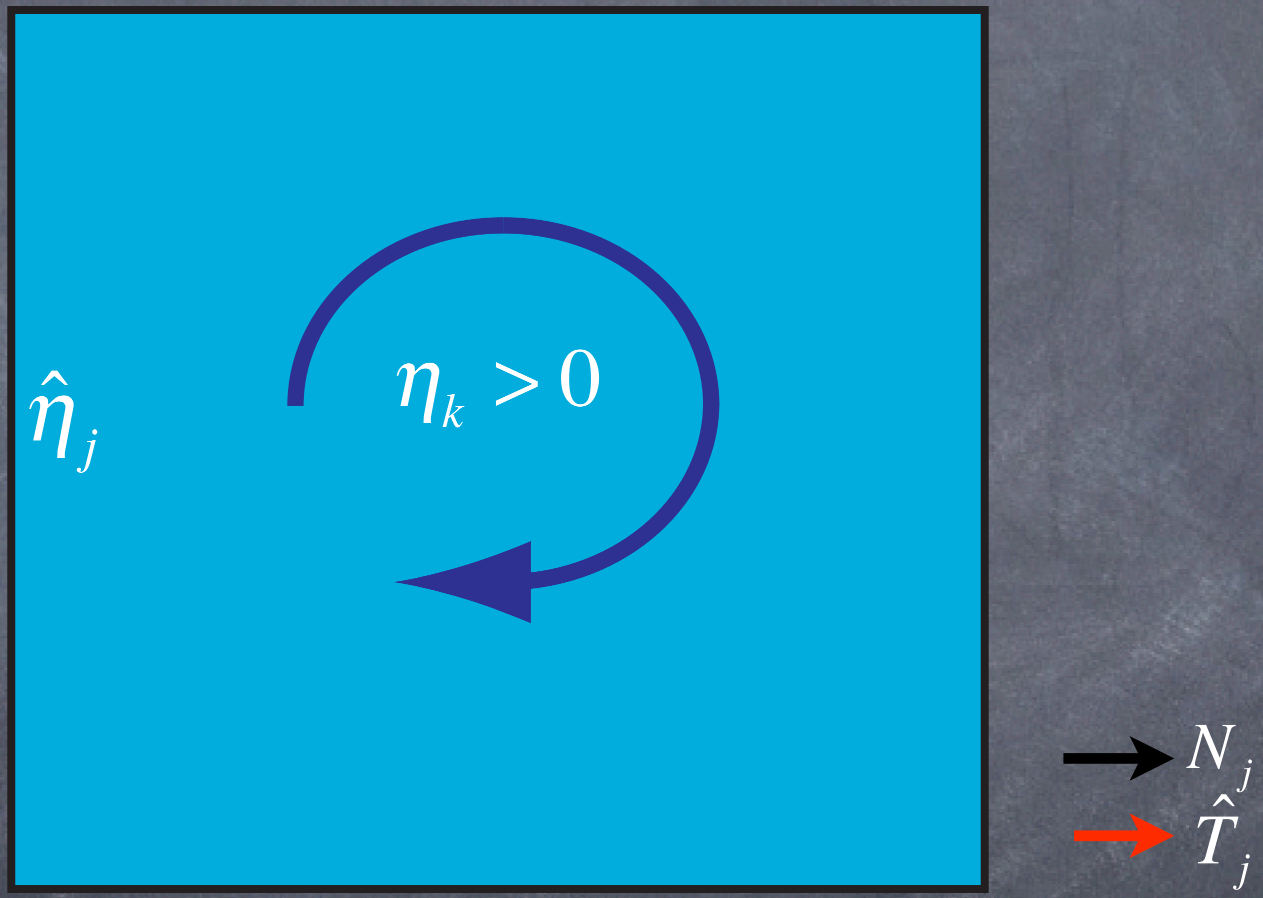
Compare to its continuous counterpart:

$$\frac{\partial \eta}{\partial t} + \nabla \bullet (\eta \underset{\sim}{u}) = 0$$



Recall the derivation of the circulation theorem

$$\int_A \frac{\partial (f+\omega)}{\partial t} dA = -\oint_C (f+\omega)\,\underset{\sim}{u} \bullet d\underset{\sim}{n}$$



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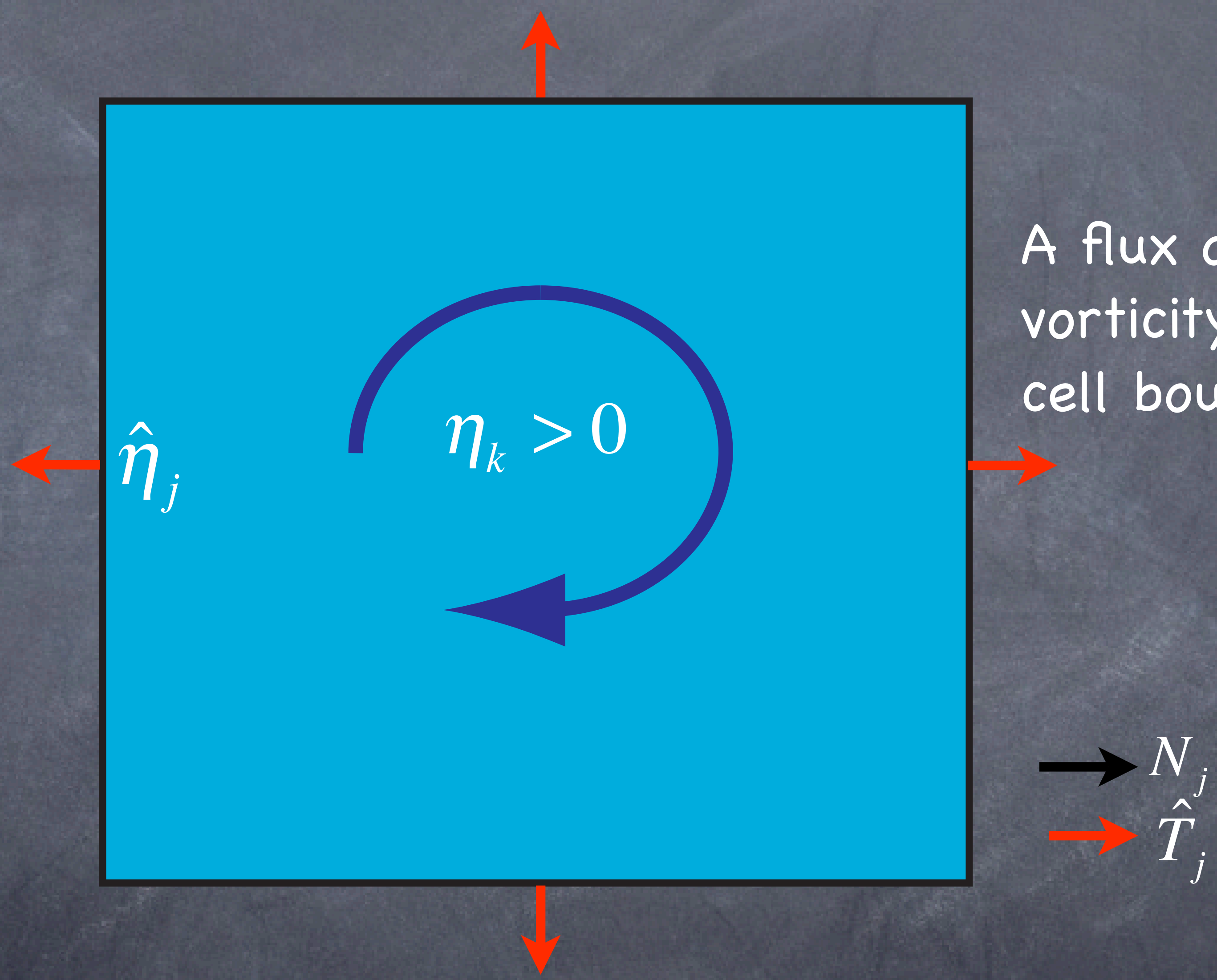
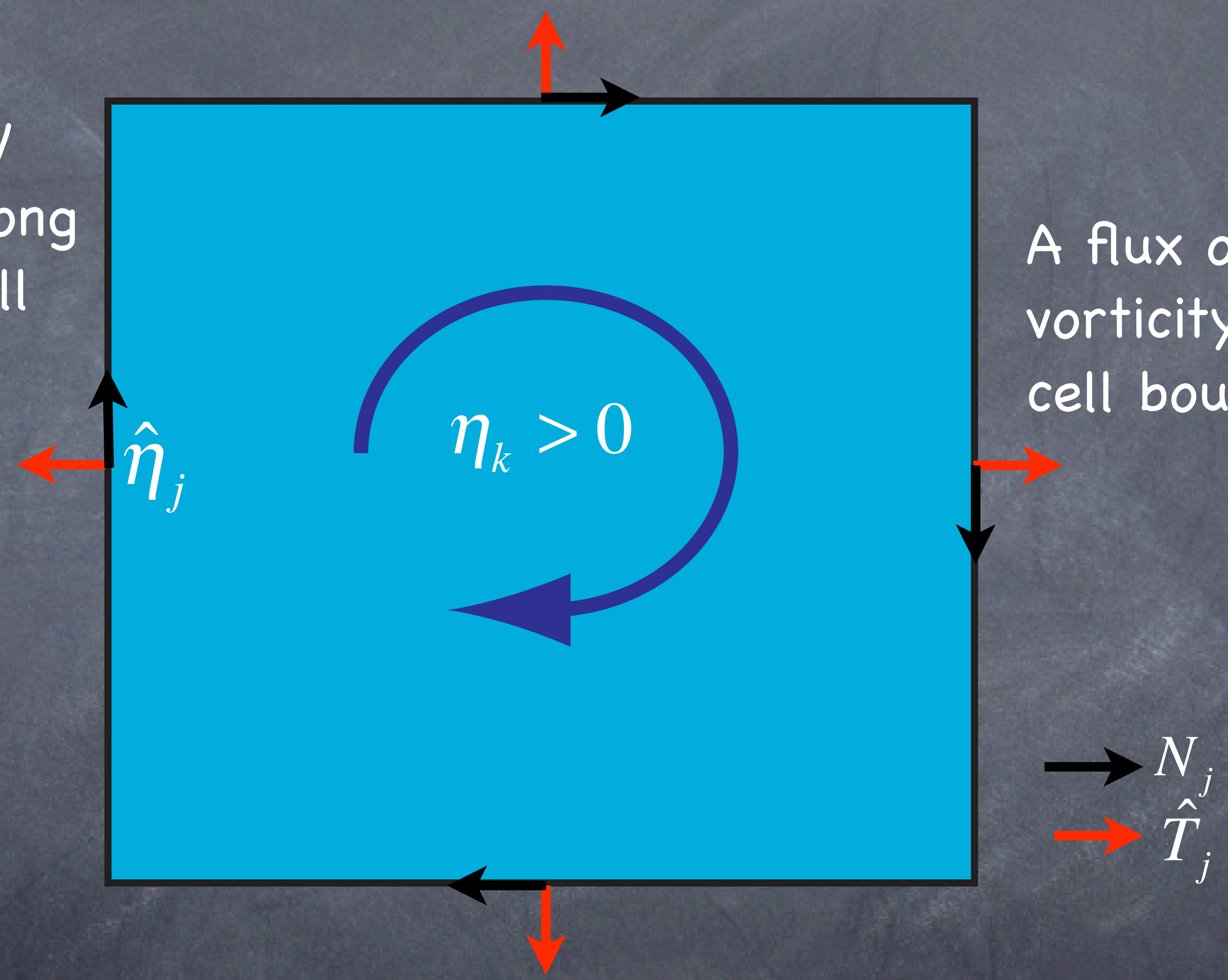


A flux of positive vorticity across the cell boundary ....

Recall the derivation of the circulation theorem

$$\int_A \frac{\partial(f+\omega)}{\partial t} dA = -\oint_C (f+\omega)\underset{\sim}{u} \bullet d\underset{\sim}{n}$$

Induces a CCW acceleration along each of the cell edges.



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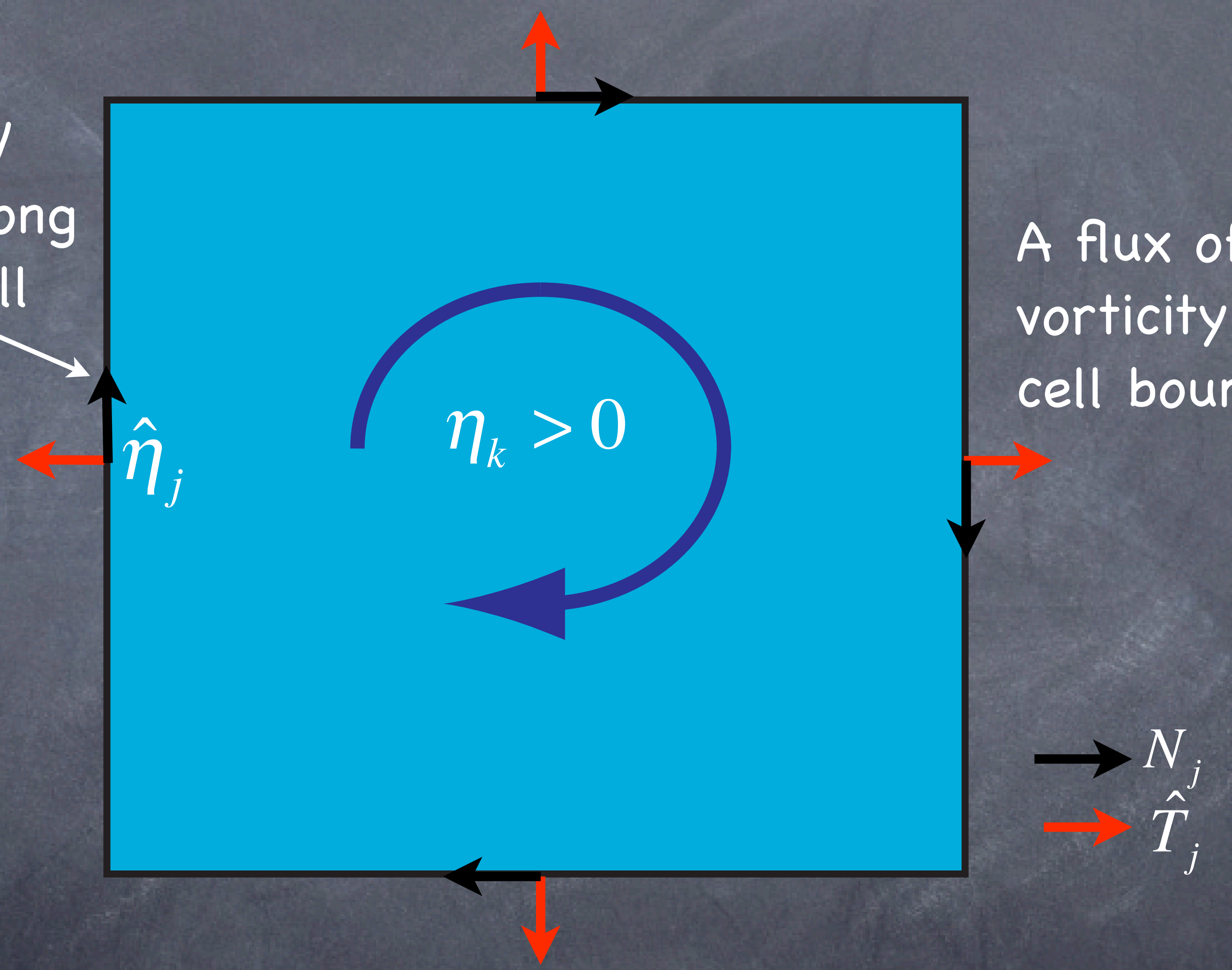


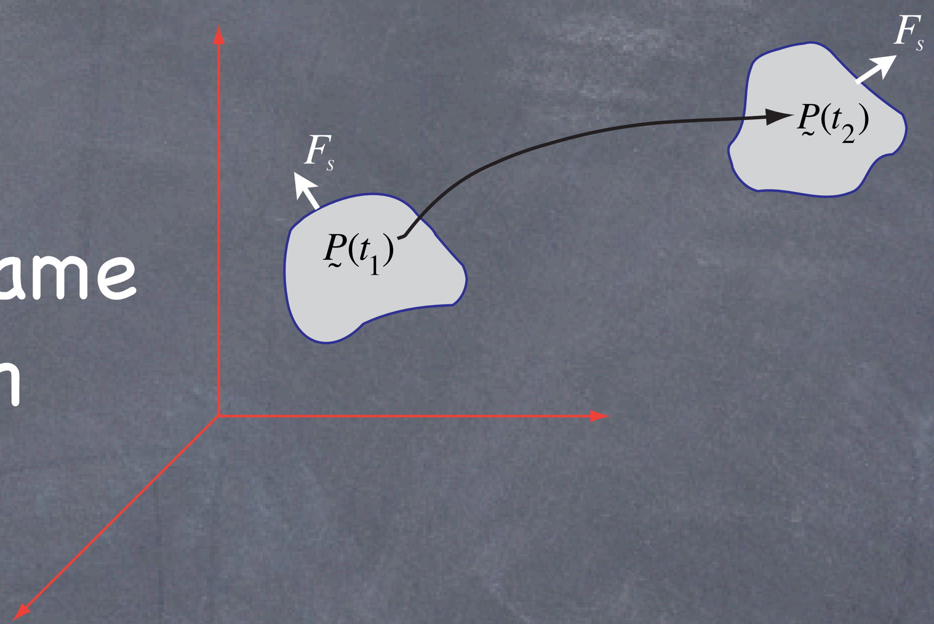
A flux of positive vorticity across the cell boundary ....

$$\frac{\partial \eta_k}{\partial t} + \frac{1}{A_k} \sum_{j=1}^{nedges} \hat{\eta}_j \hat{T}_j dc_j = 0$$

: our discrete vorticity equation obeys this principle and holds for any closed loop in the domain. Note that $\hat{\eta}_j$ is still undetermined (this is the topic for Friday).

Summary, 1 of 3

We developed evolution equations for mass, momentum and circulation in a reference frame following a set of particles, i.e. a Lagrangian reference frame.



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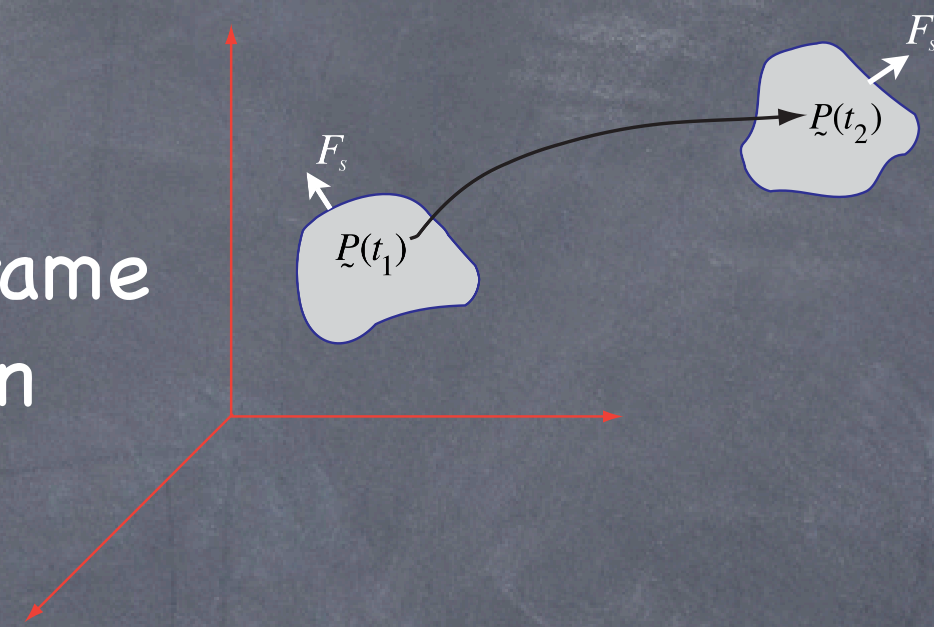
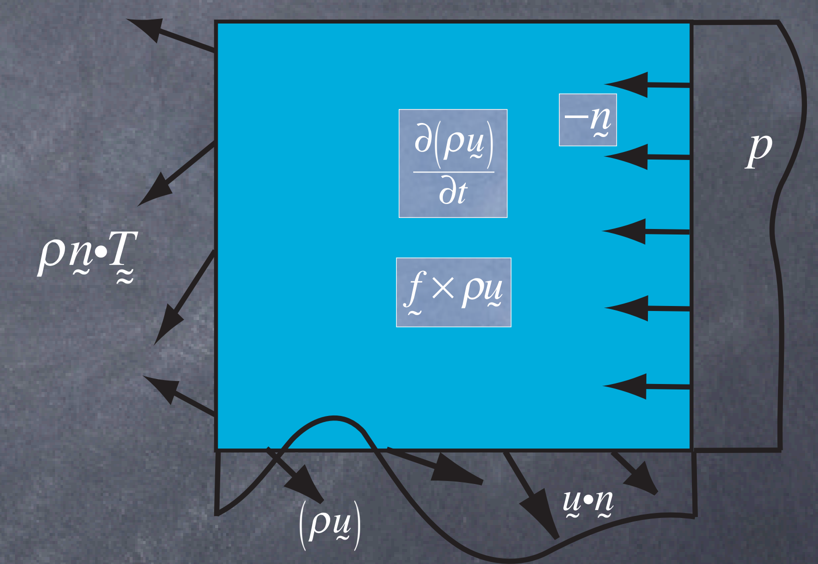
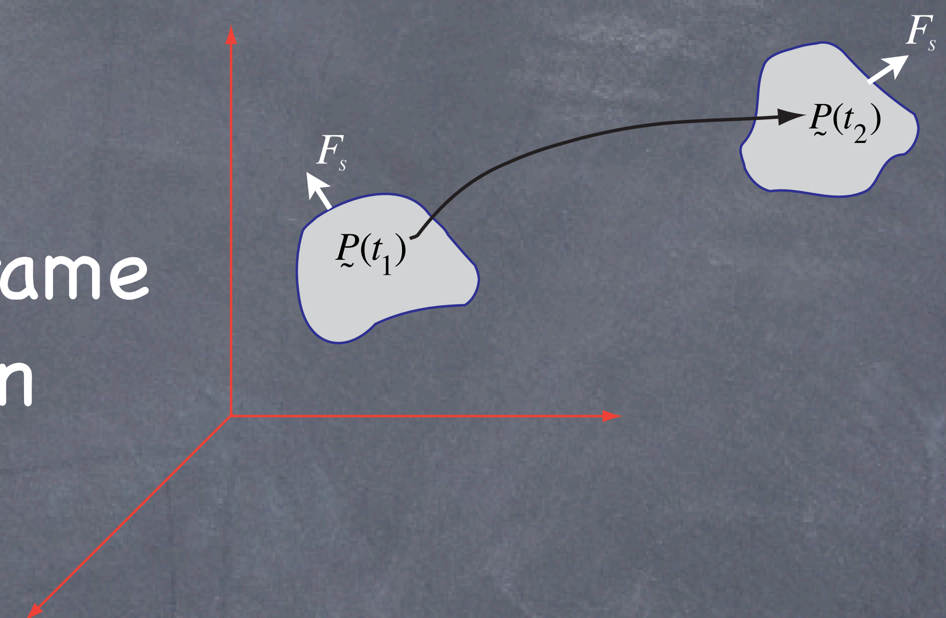


Using Reynold's Tranport Theorem, we recast these evolution equations in a fixed, i.e. Eulerian, reference frame that is better suited to long-time integrations.



Summary, 1 of 3



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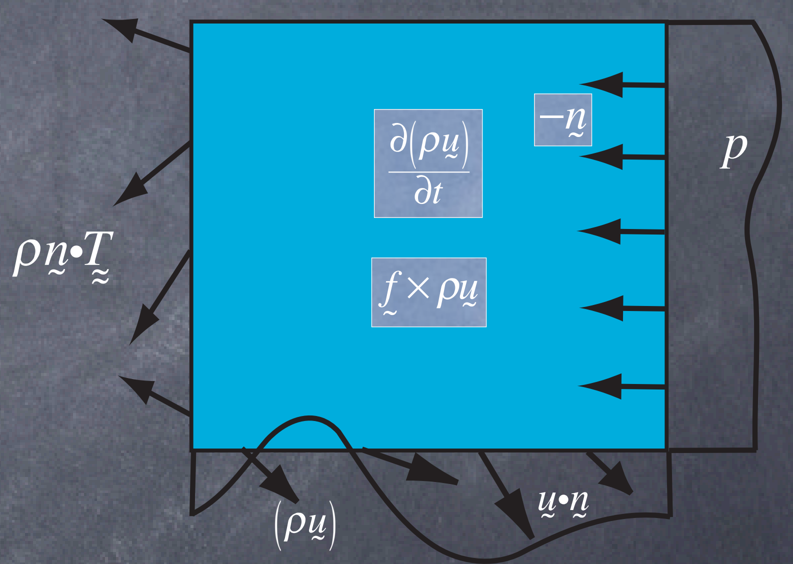

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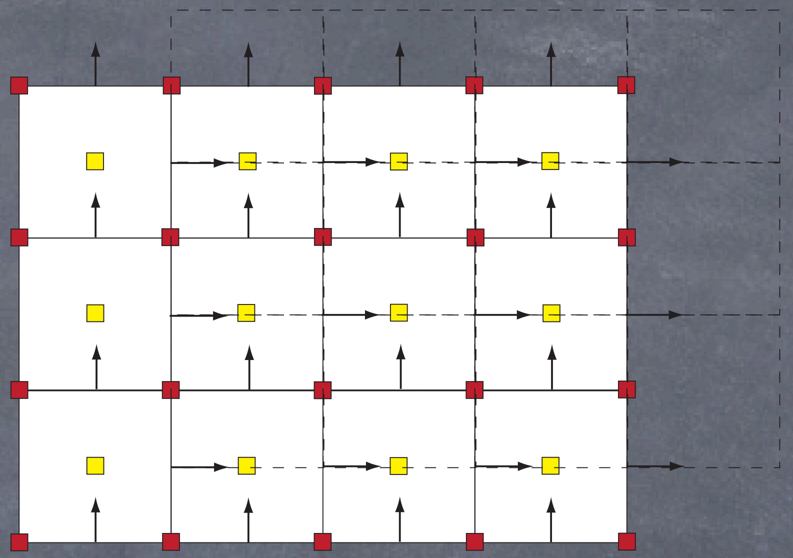
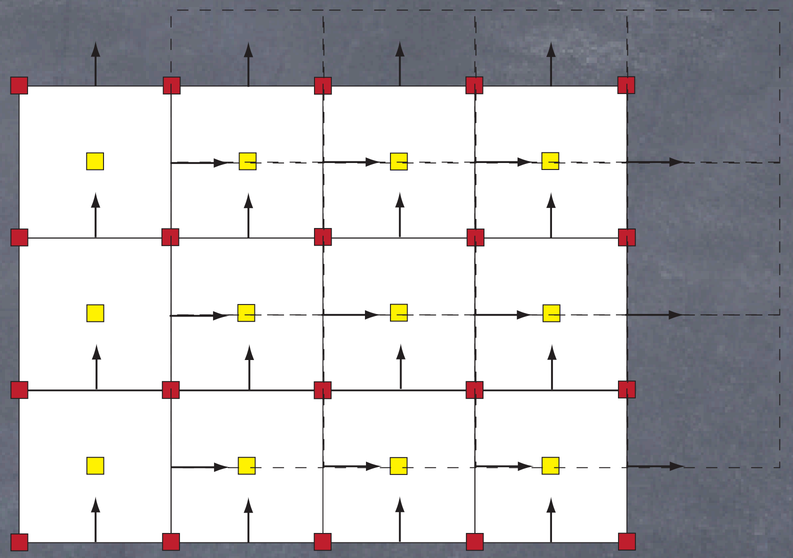
We developed various forms of F=ma, each with its own advantages and disadvantages: advective, flux-form, invariant, vor/div form.

Summary, 2 of 3

We chose to focus on the invariant/C-grid combination to develop a discrete model of F=ma.



Summary, 2 of 3



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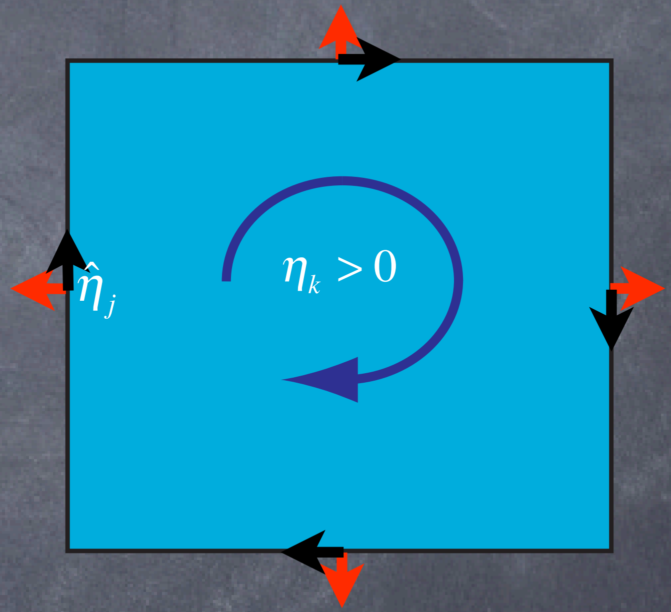

We derived the discrete counterpart to Kelvin's circulation theorem in this system.

Segues into the next three lectures:

Summary, 3 of 3

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The discrete derivation developed above holds (unchanged) for any mesh that is locally orthogonal, including triangles, quads and hexagons. The relative merits of each of these meshes will be discussed on Tuesday.

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Summary, 3 of 3

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I asserted that the discrete form of F=ma developed above offers precise control over the evolution of the vorticity equation (essentially as precise as prognosing the vorticity field itself). This will be more fully developed on Thursday when we discuss transport, monotonicity and how to determine $\hat{\eta}_j$.

What is missing?

I motivated the importance of F=ma with three reasons:

1) the balance of forces is important.

2) the evolution of F=ma determines the vorticity field, and vorticity is of primary concern to us.

3) the evolution of F=ma determines the kinetic energy field, which in turn participates in the system energetics (i.e. the flow of energy between its kinetic, potential and internal forms).

Hopefully I have done justice to #1 and #2, but I have completely omitted #3. Bonaventura and Ringler, 2005, MWR, vol 133, pg 2351 discuss energetics (KE) in the context of the discrete method developed above.