Energy, Geostrophic Adjustment, and a Momentum Analog to the Z-grid

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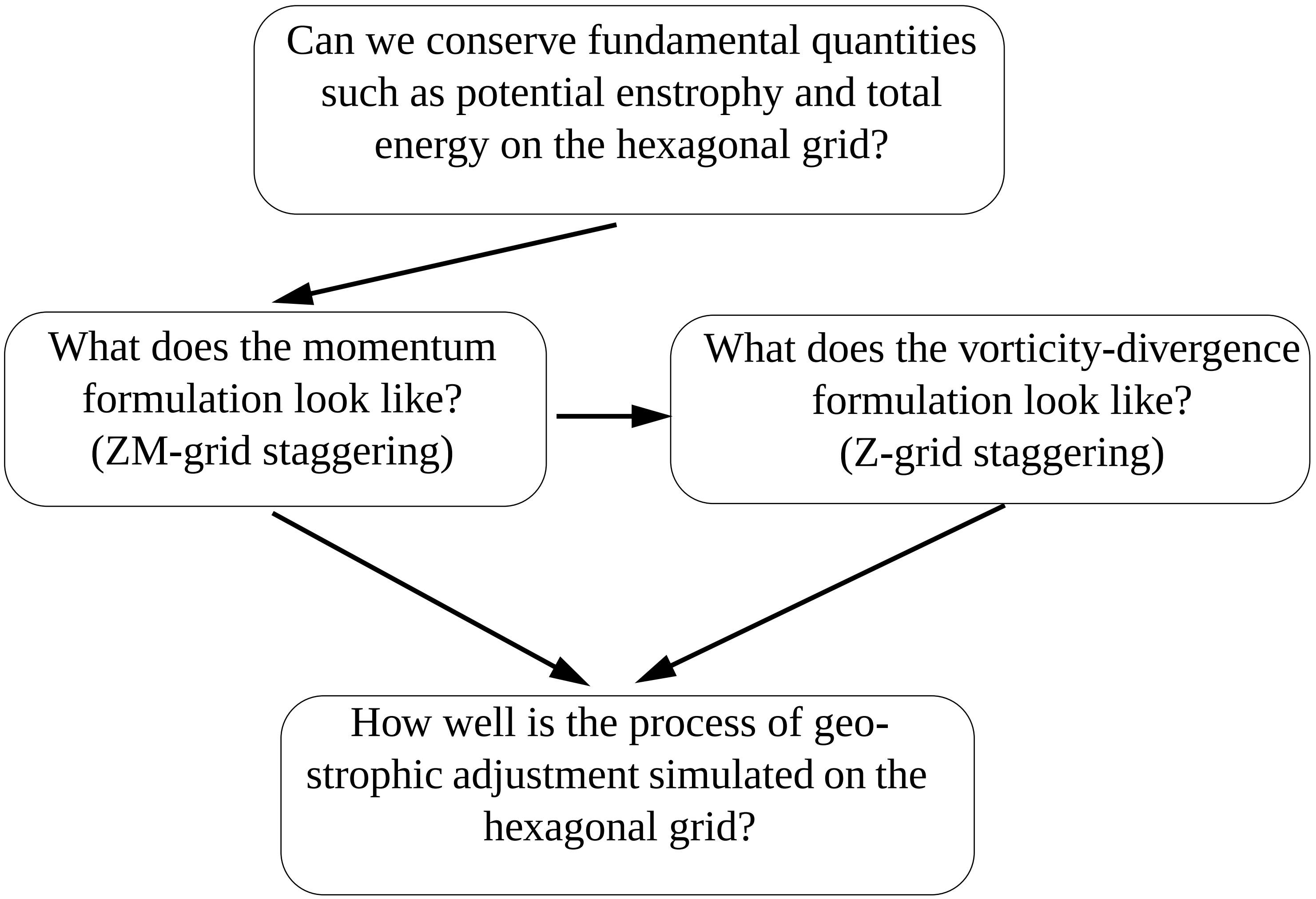
The Basic Questions



The ZM-grid staggering

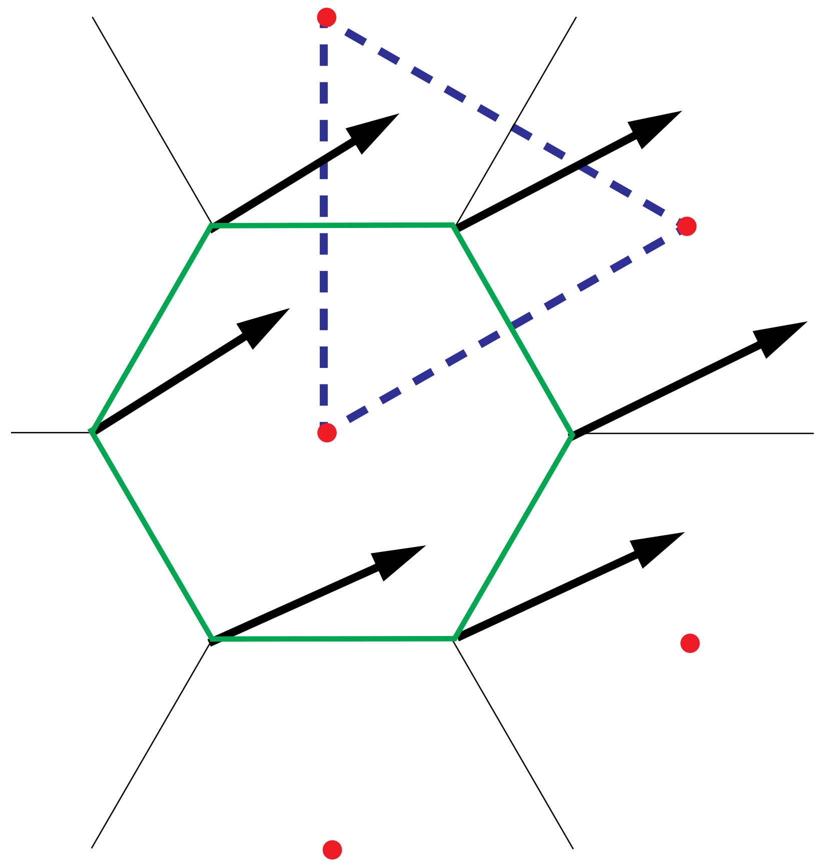
All scalars at grid cell centers
- Mass
- Kinetic Energy
- Vorticity
- Divergence

All vectors at grid cell corners
- Velocity
- Gradients of scalar fields

area of each momentum point

is $\frac{1}{2}$

the area of each mass point



The Shallow Water Equations

Continuous Equations

$$\frac{\partial h}{\partial t} = -\nabla \bullet (h\underset{\sim}{V})$$

$$\frac{\partial}{\partial t}\underset{\sim}{V} = -\eta \underset{\sim}{k} \times \underset{\sim}{V} - \nabla[K + gh]$$

Discrete Equations

$$\frac{\partial h_0}{\partial t} = -D_0[\bar{h}_c \underset{\sim}{V}_c]$$

$$\frac{\partial \underset{\sim}{V}_c}{\partial t} = -\bar{\eta}_c \underset{\sim}{k} \times \underset{\sim}{V}_c - \underset{\sim}{G}_c[K_0 + gh_0]$$

D_0 = discrete divergence operator

$\underset{\sim}{G}_c$ = discrete gradient operator

$\bar{h}_c$ = averaging of mass to cell corners

$\bar{\eta}_c$ = averaging of vorticity to cell corners

C_0 = discrete curl operator

Discrete Divergence and Curl Operators

Definition

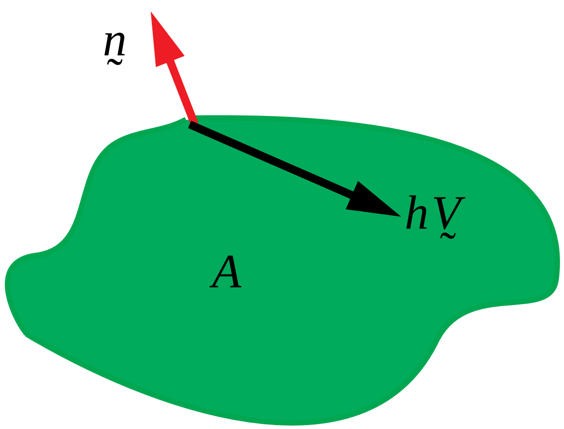
$$\nabla \bullet (h\underset{\sim}{V}) \equiv \lim_{A \to 0} \frac{1}{A} \int \underset{\sim}{n} \bullet h\underset{\sim}{V} \; dl$$

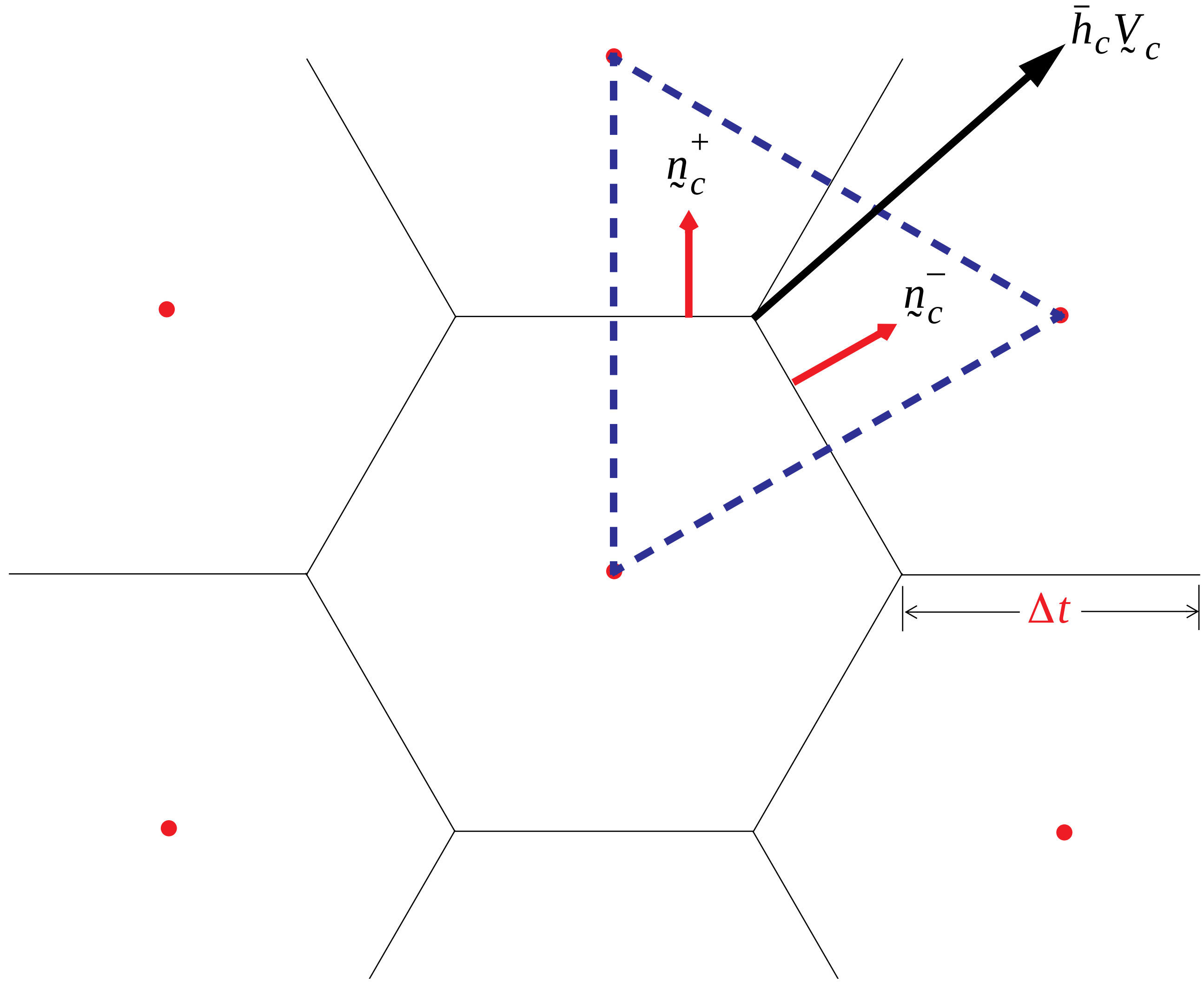
$$\nabla \times (h\underset{\sim}{V}) \equiv \lim_{A \to 0} \frac{1}{A} \int \underset{\sim}{n} \times h\underset{\sim}{V} \; dl$$

By Analogy

$$D_0(h\underset{\sim}{V}) = \frac{1}{A_0} \sum_{c=1}^{6} (\underset{\sim}{n}_c^{+} + \underset{\sim}{n}_c^{-}) \bullet \bar{h}_c \underset{\sim}{V}_c \left(\frac{\Delta t}{2}\right)$$

$$C_0(h\underset{\sim}{V}) = \frac{1}{A_0} \sum_{c=1}^{6} (\underset{\sim}{n}_c^{+} + \underset{\sim}{n}_c^{-}) \times \bar{h}_c \underset{\sim}{V}_c \left(\frac{\Delta t}{2}\right)$$





$\bar{h}_c \underset{\sim}{V}_c$
$\underset{\sim}{n}_c^+$
$\underset{\sim}{n}_c^-$
Δt

Conservation of Total Energy

Continuous Equations

$$K\left\{\frac{\partial h}{\partial t} = -\nabla\bullet(h\underset{\sim}{V})\right\}$$

$$gh\left\{\frac{\partial h}{\partial t} = -\nabla\bullet(h\underset{\sim}{V})\right\}$$

$$h\underset{\sim}{V}\bullet\left\{\frac{\partial}{\partial t}\underset{\sim}{V} = -\eta\underset{\sim}{k}\times\underset{\sim}{V} - \nabla[K+gh]\right\}$$

$$\int_A h\left[K+\frac{1}{2}gh\right]dA = 0$$

Discrete Equations

$$K_0\left\{\frac{\partial h_0}{\partial t} = -D_0[\bar{h}_c, \underset{\sim}{V}_c]\right\}$$

$$gh_0\left\{\frac{\partial h_0}{\partial t} = -D_0[\bar{h}_c, \underset{\sim}{V}_c]\right\}$$

$$h_0\sum_{c=1}^{6}\underset{\sim}{V}_c\bullet\left\{\frac{\partial \underset{\sim}{V}_c}{\partial t} = -\bar{\eta}_c\underset{\sim}{k}\times\underset{\sim}{V}_c - \underset{\sim}{G}_c[K_0+gh_0]\right\}$$

$$\sum_{i=0}^{n} h_i\left[K_i+\frac{1}{2}gh_i\right]A_i = 0$$

We use our degrees of freedom in G_c, K_0, and $\bar{h}_c$ to make this happen.

So what the are forms of $\underset{\sim}{G}_c$, K_0, and $\bar{h}_c$

The Gradient Operator

$$\underset{\sim}{G}_c(K) \bullet \underset{\sim}{e}_1 = \frac{\Delta n}{A_c}\left[\left(\frac{K_1 + K_2}{2}\right) - K_0\right]$$

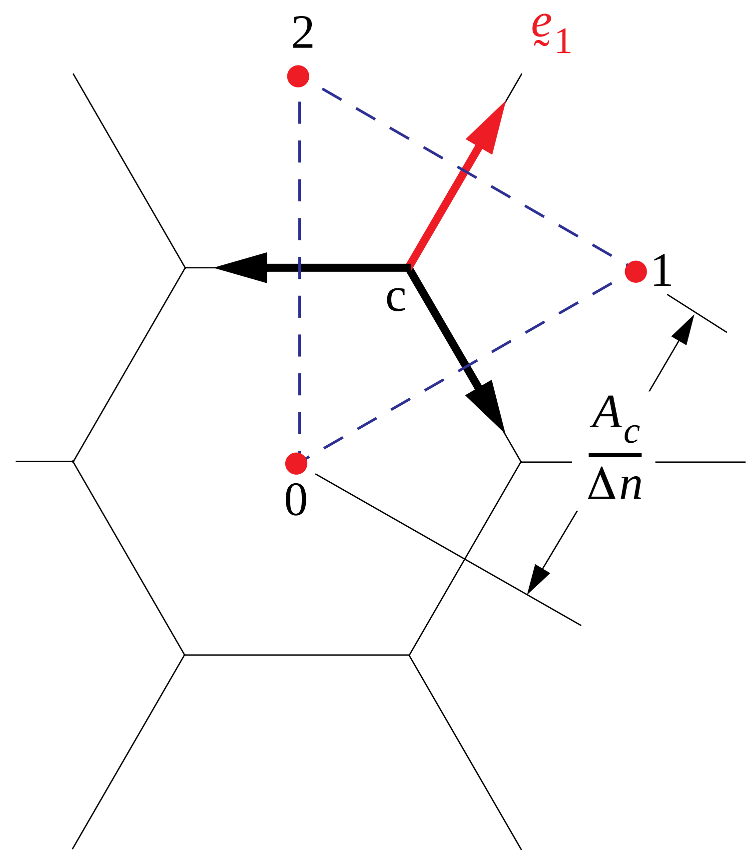
other components look similar

Kinetic Energy

$$K_0 = \frac{1}{6}\sum_{c=1}^{6}\frac{1}{2}(\underset{\sim}{V}_c \bullet \underset{\sim}{V}_c)$$

The Mass Averaging Operator

$$\bar{h}_c = \frac{1}{3}(h_0 + h_1 + h_2)$$



A 2-D turbulence simulation

Momentum formulation of full shallow-water equations with free surface

doubly-periodic f-plane, $f = 1.0 \times 10^{-4} s^{-1}$

Initial Conditions: $h = 400 \pm 50m$, $\delta = \pm 5.0 \times 10^{-5} s^{-1}$, $\zeta = \pm 5.0 \times 10^{-5} s^{-1}$

Resolution: 128x128 grid, $\Delta n = 100km$, $\lambda = 700km$

Simulation 1

Purpose: address conservation

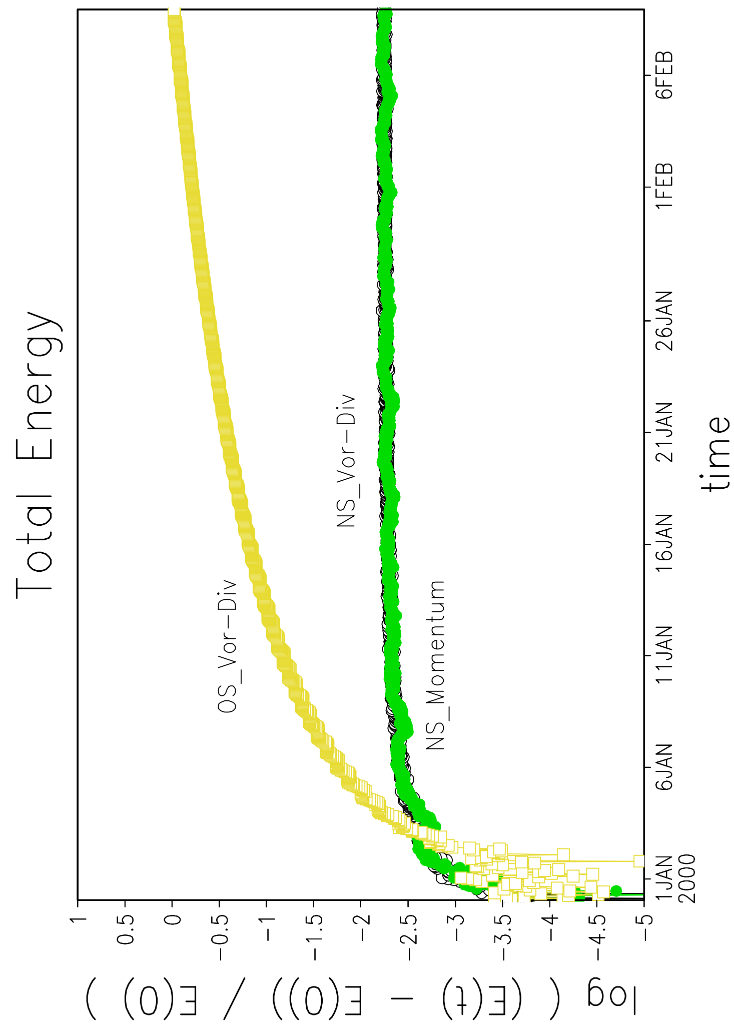
integration length 40 days
no dissipation

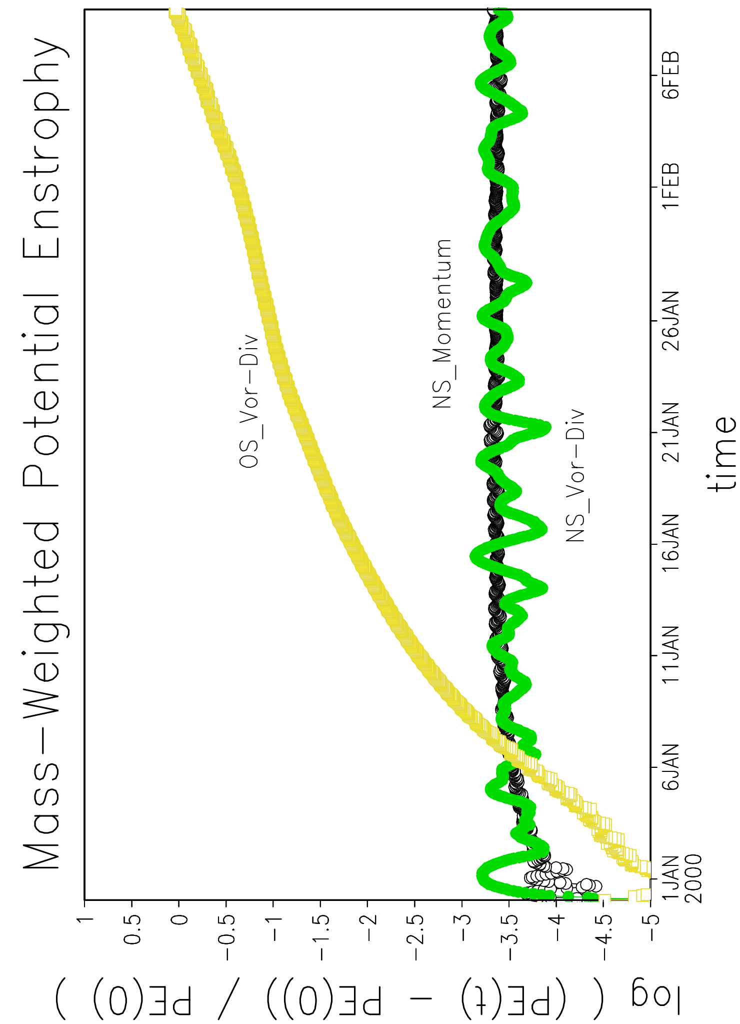
Simulation 2

Purpose: Energy and Enstrophy Cascade

integration length 2000 days

∇^6 diffusion on velocity vector


Total Energy
OS_Vor−Div
NS_Vor−Div
NS_Momentum
log ((E(t) − E(0)) / E(0))
time
1JAN 2000
6JAN
11JAN
16JAN
21JAN
26JAN
1FEB
6FEB


Mass−Weighted Potential Enstrophy
OS_Vor−Div
NS_Momentum
NS_Vor−Div
log ((PE(t) − PE(0)) / PE(0))
time
1JAN 2000
6JAN
11JAN
16JAN
21JAN
26JAN
1FEB
6FEB

Exchanging the Momentum Formulation for the Vorticity-Divergence Formulation

Continuous Equations

$$\nabla\bullet\left\{\frac{\partial}{\partial t}\underset{\sim}{V} = -\eta\underset{\sim}{k}\times\underset{\sim}{V} - \nabla[K + gh]\right\}$$

$$\nabla\times\left\{\frac{\partial}{\partial t}\underset{\sim}{V} = -\eta\underset{\sim}{k}\times\underset{\sim}{V} - \nabla[K + gh]\right\}$$

$$\downarrow$$

$$\frac{\partial\delta}{\partial t} = \underset{\sim}{k}\bullet\nabla\times(\eta\underset{\sim}{V}) - \nabla^2[K + gh]$$

$$\frac{\partial\eta}{\partial t} = -\nabla\bullet(\eta\underset{\sim}{V})$$

Discrete Equations

$$D_0\left\{\frac{\partial\underset{\sim}{V}_c}{\partial t} = -\bar{\eta}_c\underset{\sim}{k}\times\underset{\sim}{V}_c - \underset{\sim}{G}_c[K_0 + gh_0]\right\}$$

$$C_0\left\{\frac{\partial\underset{\sim}{V}_c}{\partial t} = -\bar{\eta}_c\underset{\sim}{k}\times\underset{\sim}{V}_c - \underset{\sim}{G}_c[K_0 + gh_0]\right\}$$

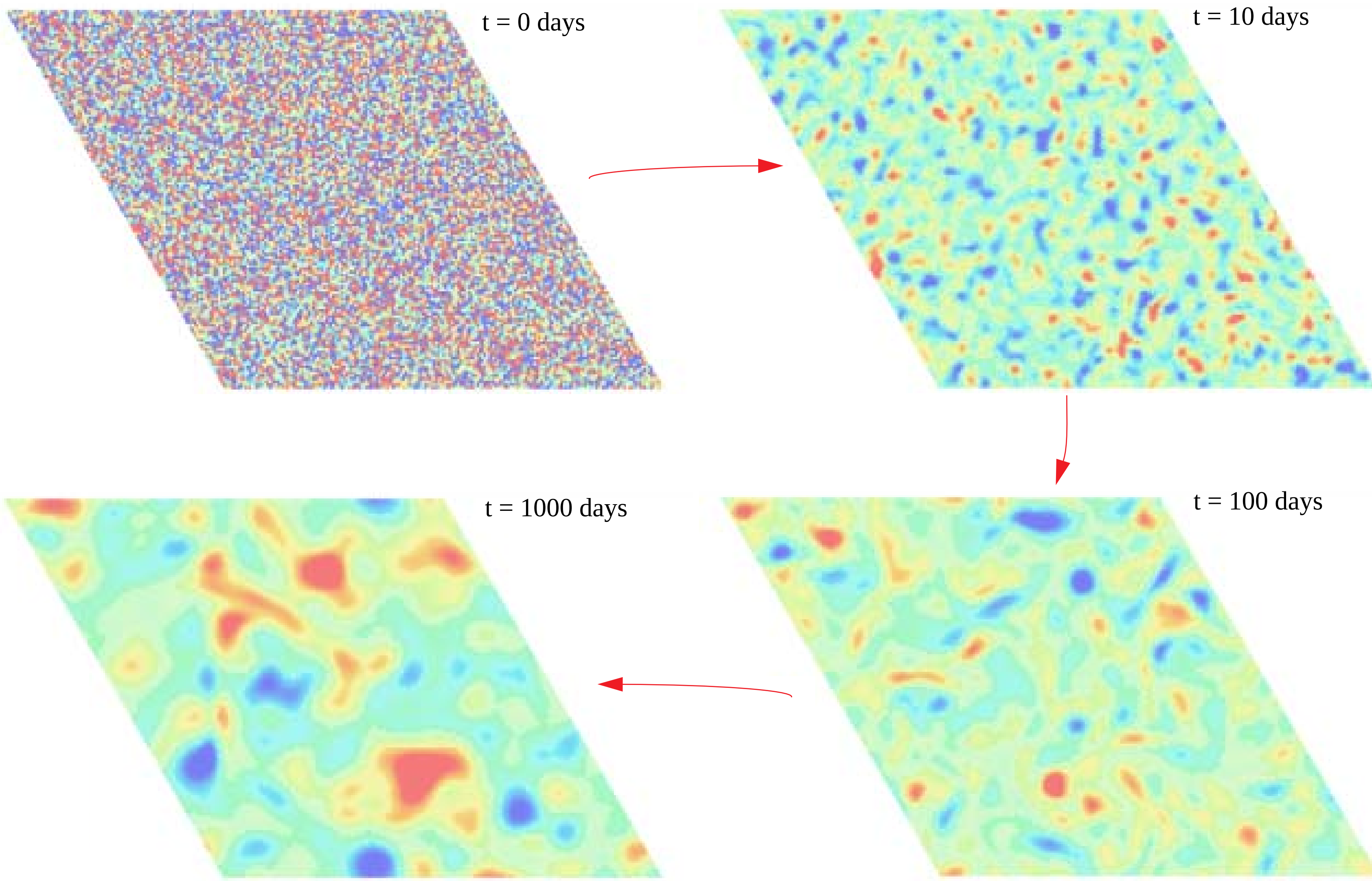
$$\downarrow$$

$$\frac{\partial\delta_0}{\partial t} = C_0(\eta\underset{\sim}{V}) - L_0[K + gh]$$

$$\frac{\partial\eta_0}{\partial t} = -D_0(\eta\underset{\sim}{V})$$

What does the discrete Laplacian look like?

Evolution of the vorticity field



So the nonlinear aspects, namely advection, checkout. What about the linear system?

Why look at the linearized equations?

1) simulation of physical modes related to geostrophic adjustment
2) isotropy
3) check for the existence of computational modes

Momentum Formulation

$$\frac{\partial h_0}{\partial t} + H(\nabla \bullet \underset{\sim}{V})_0 = 0$$

$$\frac{\partial \underset{\sim}{V}_c}{\partial t} + f k \times \underset{\sim}{V}_c + g(\nabla h)_c = 0$$

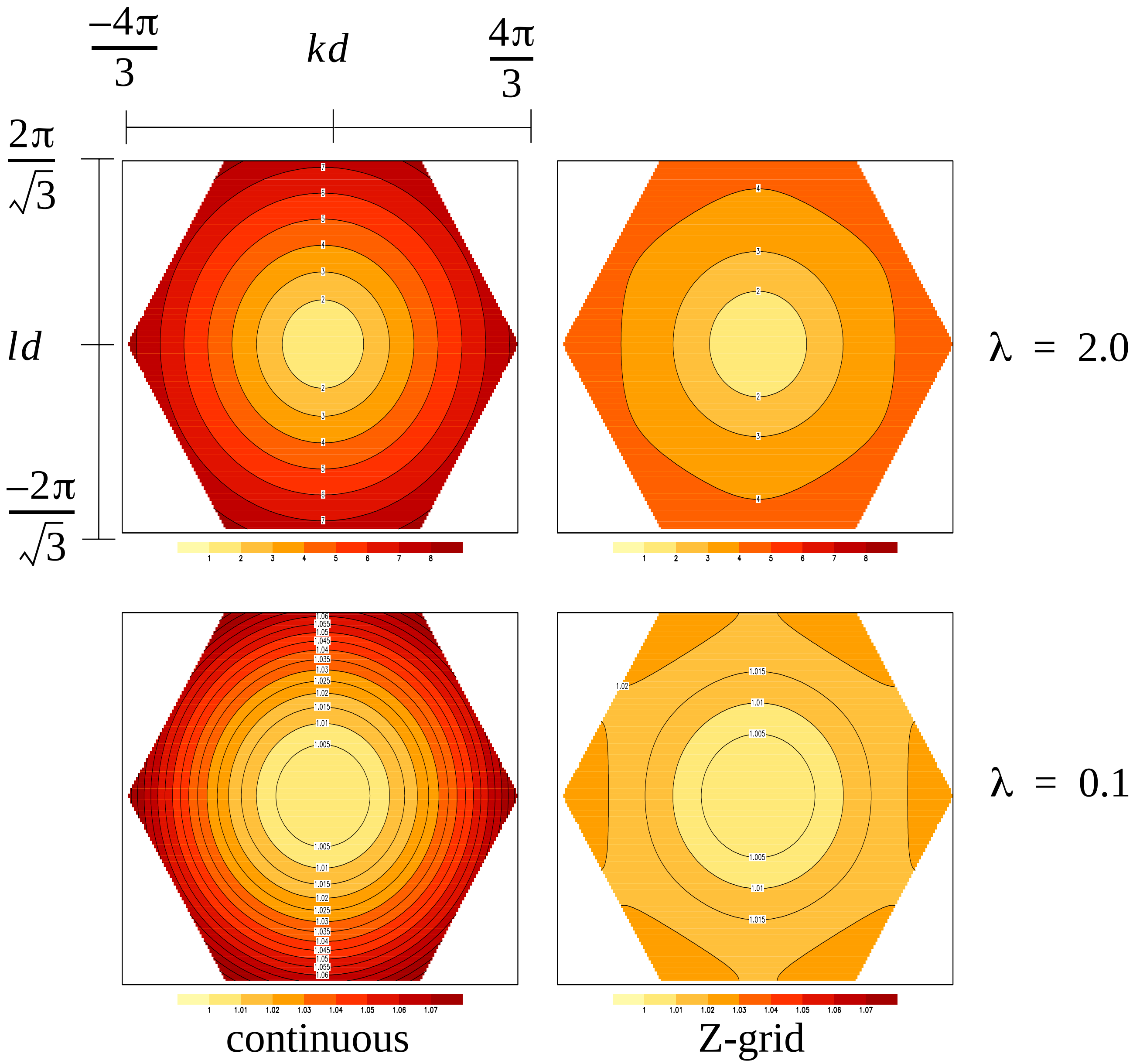
Vorticity-Divergence Formulation

$$\frac{\partial h_0}{\partial t} + H\delta_0 = 0$$

$$\frac{\partial \delta_0}{\partial t} - f\zeta_0 + g(\nabla^2 h)_0 = 0$$

$$\frac{\partial \zeta_0}{\partial t} + f\delta_0 = 0$$

The three modes found in the Vorticity-Divergence formulation are modes in the Momentum formulation.



$\frac{-4\pi}{3}$
kd
$\frac{4\pi}{3}$
$\frac{2\pi}{\sqrt{3}}$
ld
$\frac{-2\pi}{\sqrt{3}}$
$\lambda = 2.0$
$\lambda = 0.1$
continuous
Z-grid

Computational modes in the ZM-grid system

Mode Facts

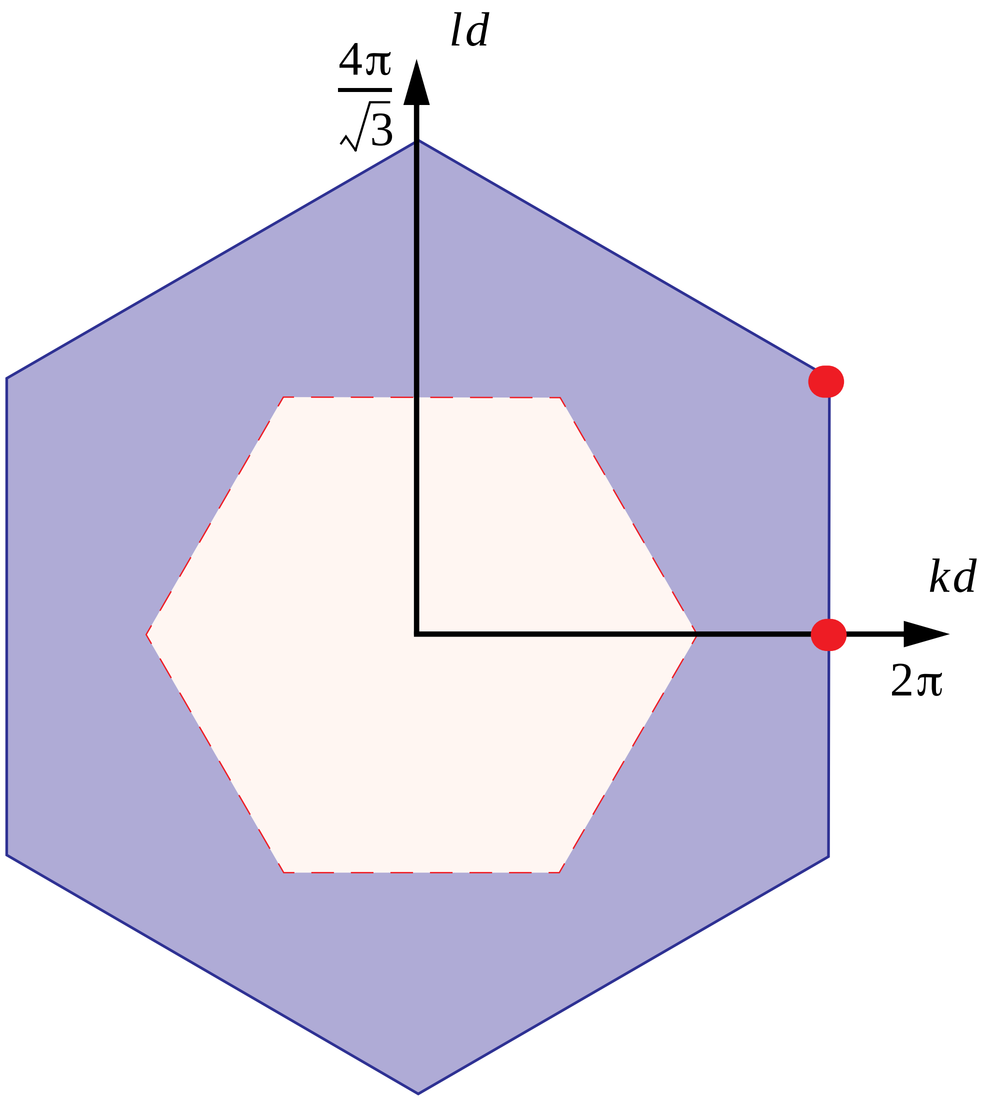
1) no modes in height field

2) two modes in velocity field

3) modes exist at wave numbers not resolved by the scalar data.

4) a 2000 day 2-D turbulence simulation required no filtering

5) we have the perfect filter



Conclusions for the vorticity-divergence (Z-grid) formulation

1) New formulation conserves potential enstrophy and total energy.

2) The isotropy of the hexagonal grid is apparent in the simulation of the geostrophic adjustment process.

3) No computational modes

Conclusions for the momentum (ZM-grid) formulation

1) New formulation conserves potential enstrophy and total energy.

2) The isotropy of the hexagonal grid is apparent in the simulation of the geostrophic adjustment process.

3) Two benign* computational modes in the velocity equation

*at least so far

Where to from here?

1) generalize to the sphere

2) test in dynamical core (predict vorticity-divergence and momentum interchangeably)

3) build an ocean model

http://kiwi.atmos.colostate.edu/BUGS/projects/geodesic

What is there?

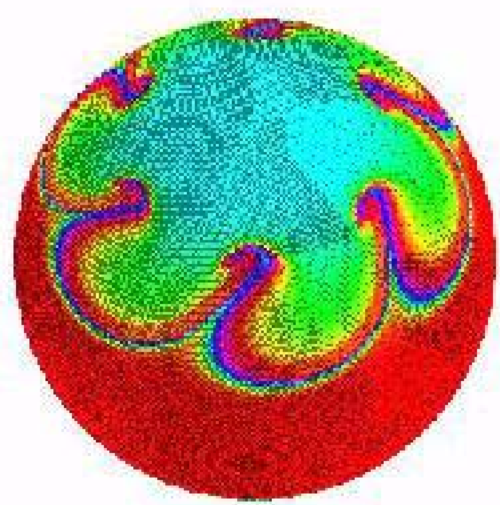
shallow water model source code

domain decomposition (MPI)

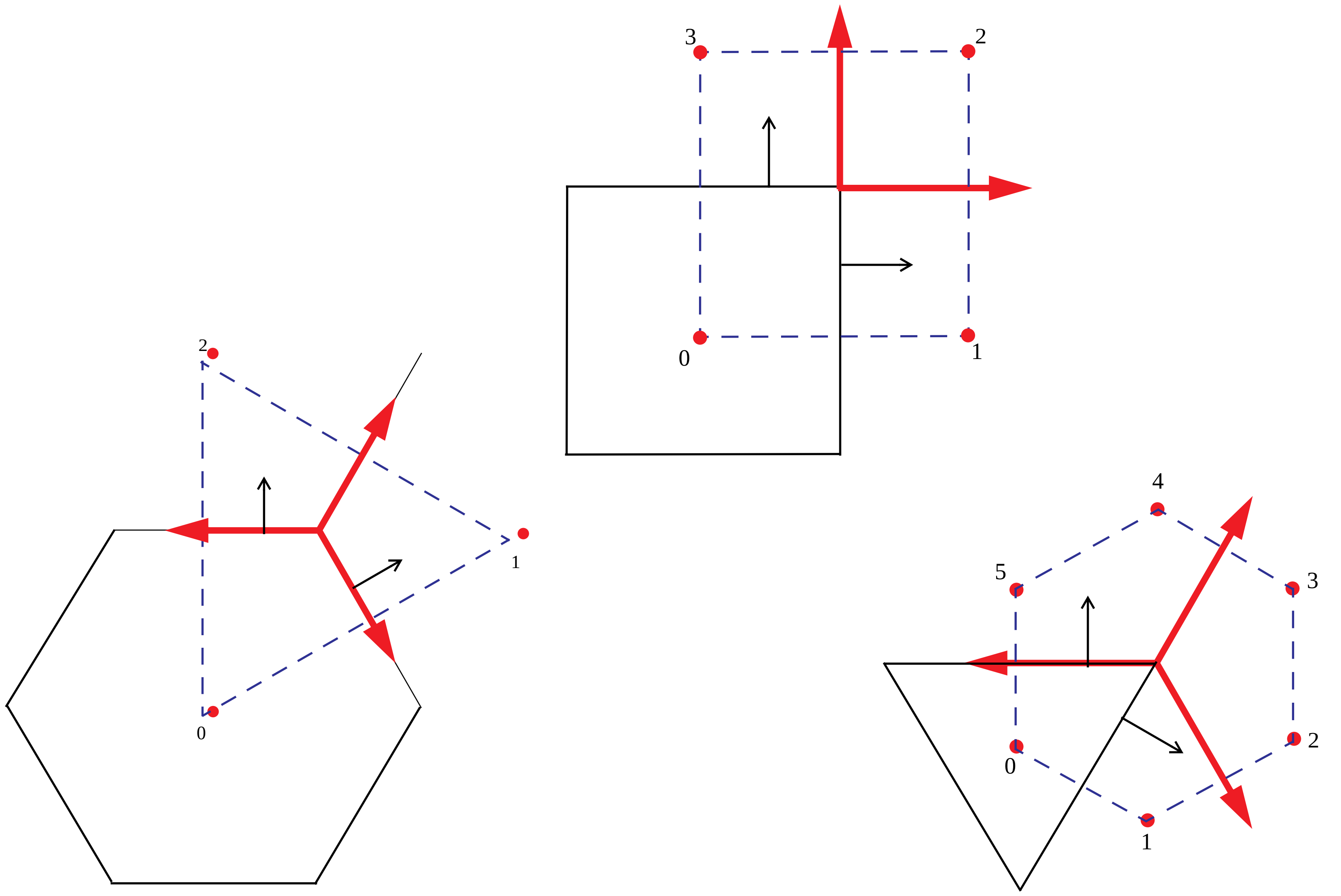
conservative remapping (SCRIP)

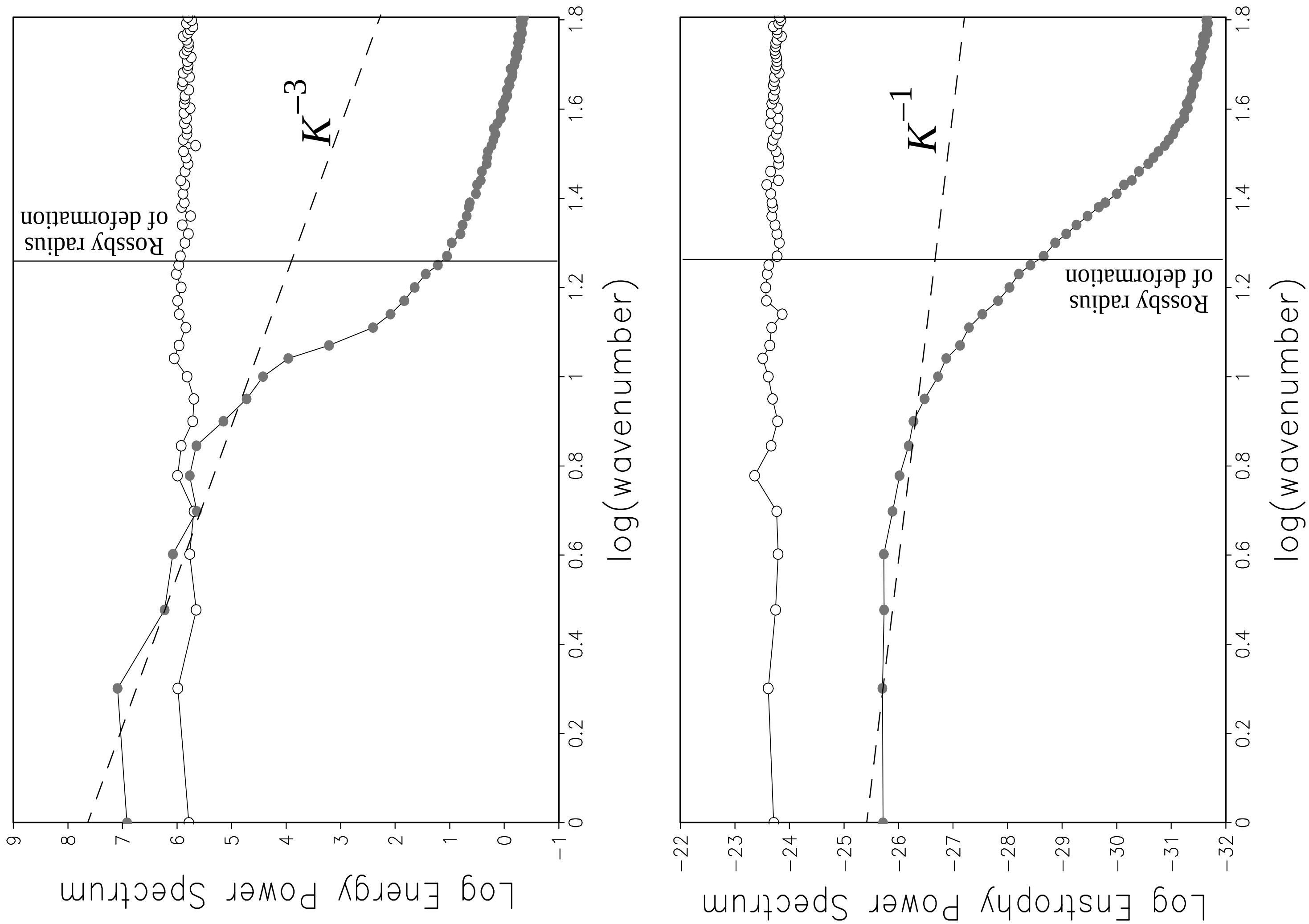
technical papers

slide show (intro to geodesic grids)

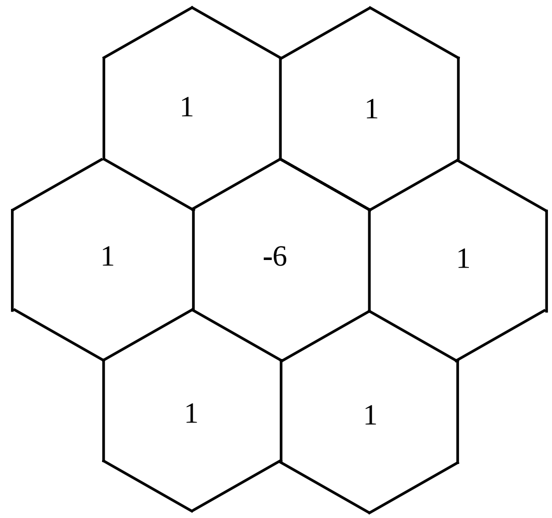


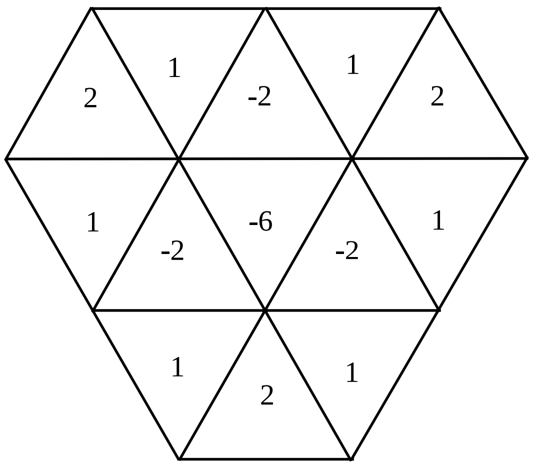
ZM-grid staggering in other geometries

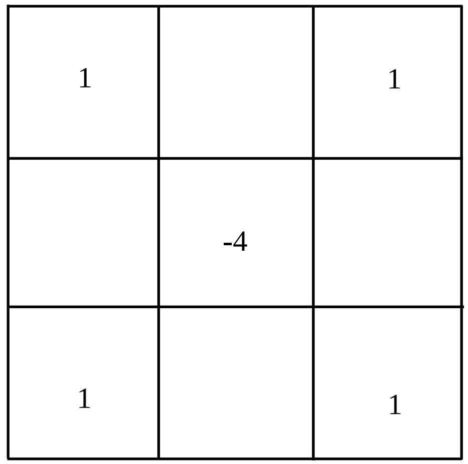


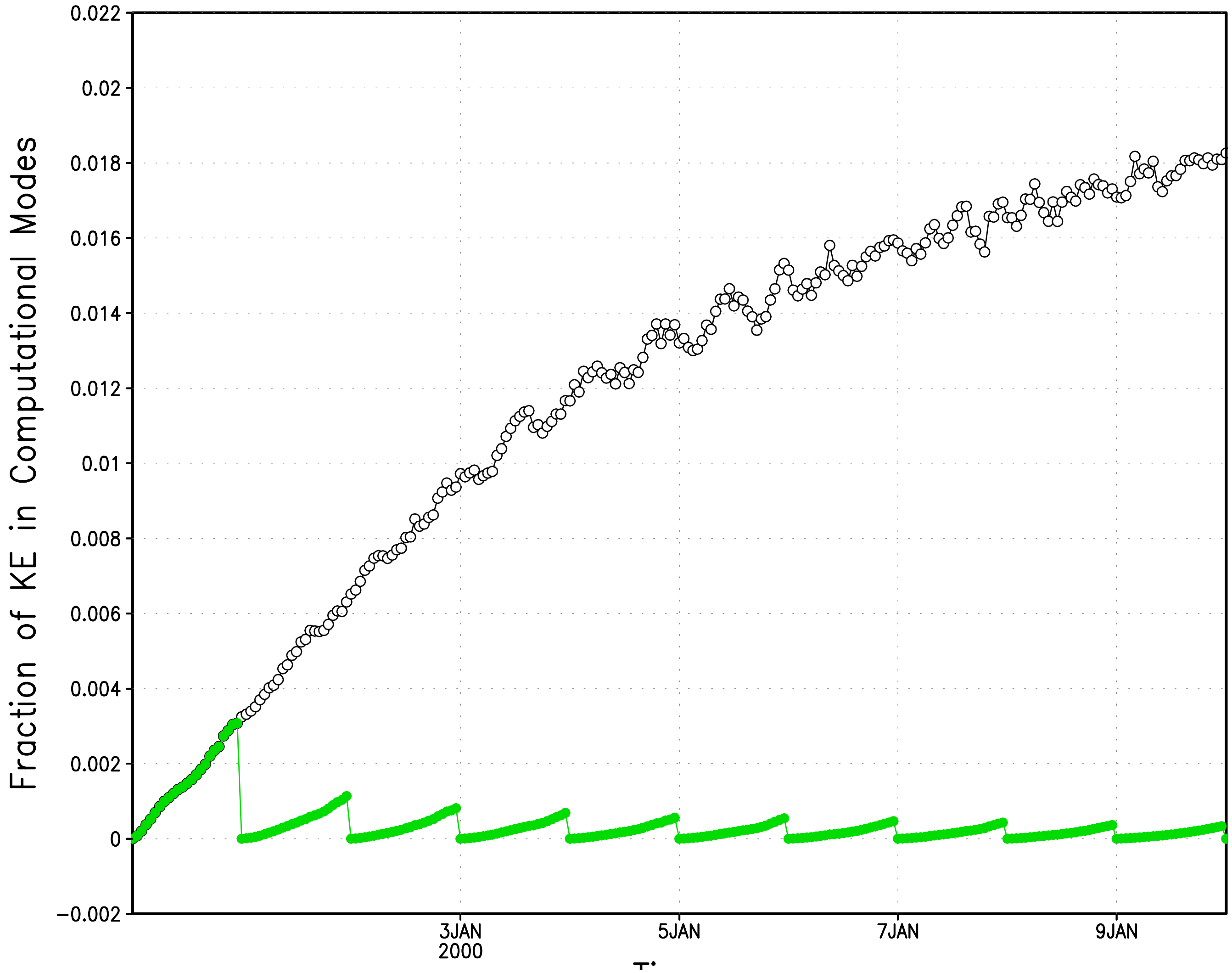

Log Energy Power Spectrum
log(wavenumber)
Rossby radius
of deformation
K^{-3}
Log Enstrophy Power Spectrum
log(wavenumber)
Rossby radius
of deformation
K^{-1}

What does the energy-conserving Laplacian look like?








Fraction of KE in Computational Modes
0.022
0.02
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
−0.002
3JAN
2000
5JAN
7JAN
9JAN

The discrete Laplacian is identical to one we are currently using.