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ABSTRACT

A scaling patch approach is used to investigate the proper scales in planar turbulent wakes. A proper scale for the mean axial flow is the well-known maximum velocity deficit $U_{\rm ref} = U_\infty - U_{\rm ctr}$, where U_∞ is the free stream velocity and $U_{\rm ctr}$ is the mean axial velocity at the wake centerline. From an admissible scaling of the mean continuity equation, a proper scale for the mean transverse flow is found as $V_{\rm ref} = (d\delta/dx)U_{\rm ref}$, where $d\delta/dx$ is the growth rate of the wake width. From an admissible scaling of the mean momentum equation, a proper scale for the kinematic Reynolds shear stress is found as $R_{uv,{\rm ref}} = U_\infty V_{\rm ref}$, which is a mixed scale of the free stream velocity and the mean transverse flow scale. Expressions are derived for the scaled mean transverse velocity and Reynolds shear stress in the far field of planar turbulent wakes. Using a Gaussian function for the mean axial velocity deficit, approximate functions for the scaled mean transverse velocity and Reynolds shear stress are developed and found to agree well with experimental and simulation data. This work reveals that the mean transverse flow, despite its small magnitude, plays an important role in the scaling and understanding of the planar turbulent wake.

I. INTRODUCTION

Free-shear flows occur when there are no solid walls directly within flow field, for example, jets, wakes, and mixing layers. Such flows are encountered in a wide range of practical applications including flying in the air, sailing in the ocean, combustion, propulsion, atomization, and environmental flows. Better understanding of free-shear turbulence is necessary to improve the design and operation of many engineering devices. Due to its simple geometry and no wall effects, turbulent free-shear flows are also of great theoretical interest to gain insight into the fundamental properties of turbulent flows in general. Turbulent free-shear flows have been studied for over a century experimentally, theoretically, and numerically.[1–27] The purpose of this paper is to, using a relatively new scaling patch approach, determine the proper scaling of planar turbulent wakes. The configuration of a planar wake, coordinate system, and notations is shown in Fig. 1.

Gough[11] carried out a detailed study of flow over and behind a flat plate using physical experiments and numerical simulations. Using hot-wire, he measured mean flow and turbulence statistics in one symmetric and one asymmetric wake. Using a flat splitter plate with tapered trailing edge, Liu[15] performed comprehensive studies of wake flows under constant pressure, adverse pressure, and favorable pressure gradients. The effect of streamwise pressure gradient is isolated by the use of constant pressure gradients, combined with identical initial conditions. The flow field survey was conducted by using both laser Doppler anemometry and hot-wire anemometry. The experimental study was complemented by a self-similarity analysis, and a new scaling has been proposed for the Reynolds stresses.[16,18] Hickey[28] performed a direct simulation and theoretical study of sub- and supersonic wakes, and investigated three distinct wake evolution scenarios: the Kelvin–Helmholtz transition, the bypass transition in an asymmetric wake, and the initially turbulent wake. Using a variety of wake generators (including airfoil, cylinders, flat plate, screen, and solid strip), Wygnanski, Champagne, and Marasli[29] studied the effects of wake generator on the turbulent far wake. Nakayama[30] measured the mean and fluctuating velocities using pressure and hot-wire probes in the attached boundary layer and wakes of two airfoil models at a low Mach number. His results indicate that the flow around the conventional airfoil is a minor perturbation of a symmetric flat-plate flow with small wake curvature and weak viscous-inviscid interaction.

Despite decades of intensive research, a fundamental question remains open in the study of turbulent wake flows: whether a “universal self-similar” state exists in the flow far away from its initiation.

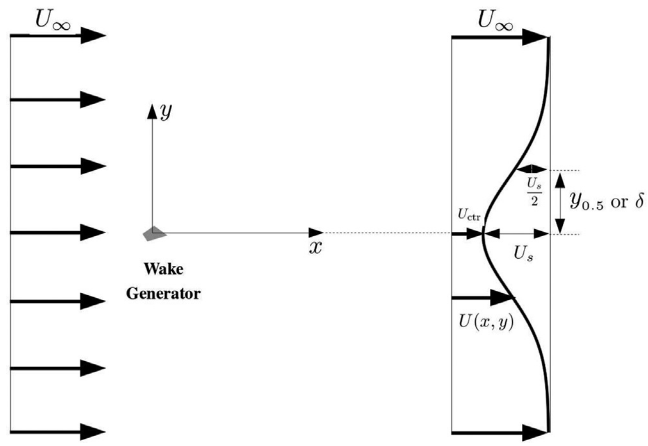

FIG. 1. Sketch of a planar wake flow initialized by a generic wake generator. x denotes the axial direction aligned along the wake centerline, and y denotes the transverse direction. U_∞ is the free stream velocity, U_{ctr} is the mean axial velocity at the wake centerline, and $U_s = U_\infty - U_{ctr}$ is the velocity deficit at the wake centerline. The wake half-width is commonly denoted as δ or $y_{0.5}$.

One hypothesis proposed by Townsend[4] is that free-shear flows become "self-similar" at a sufficient downstream distance. Empirically, the mean axial velocity deficit from different experiments and different axial locations, when normalized as $(U_\infty - U(x, y))/(U_\infty - U_{ctr}(x))$, is observed to collapse onto a single curve; that is, the U deficit profiles become self-similar. Given the direct connection among the mean axial velocity, mean transverse velocity, and Reynolds shear stress in the governing equations, one may assert that the mean transverse flow and the mean Reynolds shear stress should also become self-similar in the far field, if U profiles become self-similar.

Wygnanski, Champagne, and Marasli[6] observed that neither the normalized Reynolds shear stress profiles $R_{uv}/(U_\infty - U_{ctr})^2$ nor the velocity variance profiles $\langle uu\rangle/(U_\infty - U_{ctr})^2$ from different wake generators collapse onto a single curve. These profiles have the same shape, but the magnitudes depend on the initial conditions. The lack of collapse among $R_{uv}/(U_\infty - U_{ctr})^2$ profiles is obviously not consistent with the concept of self-similarity. To explain the findings of Wygnanski *et al.*,[6] George[7] performed a similarity analysis and proposed that there may exist a multiplicity of self-preserving states instead of one universal self-similar state.

In traditional similarity analyses,[4] the mean continuity equation is usually integrated such that the mean transverse velocity does not appear directly in the analysis process or the final results. In this work, we apply a scaling patch approach to investigate the scaling properties of planar turbulent wakes, and one goal is to determine a proper scaling for the mean transverse flow. In Sec. II, the governing equations for the mean flow are presented, and self-similar variables are defined. Proper scales for the mean transverse flow and the Reynolds shear stress are then determined by seeking admissible scaling for the mean continuity equation and the mean momentum equation. Analytical expressions for the mean transverse flow and Reynolds shear stress are also obtained. In Sec. III, the analysis results are compared with experimental data. Section IV summarizes the work.

II. SCALING PATCH ANALYSIS OF THE GOVERNING EQUATIONS

This work considers incompressible, single phase flows. Free-shear flows are "slender"; they spread slowly in the transversal direction.[1,13] Therefore, Prandtl's boundary layer equations have been used to study turbulent free-shear flows. The mean continuity equation and mean axial-momentum equation are[1,4,13,31]

$$0 = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}, \tag{1a}$$

$$0 = -U\frac{\partial U}{\partial x} - V\frac{\partial U}{\partial y} + \nu\frac{\partial^2 U}{\partial y^2} + \frac{\partial R_{uv}}{\partial y}. \tag{1b}$$

Herein, x and y are the axial and transverse coordinates, respectively. ν is the kinematic viscosity. An upper case letter denotes a mean flow variable, and a lower case letter denotes its fluctuation. For example, U is the mean velocity in the axial direction, and u is the velocity fluctuation in the axial direction. V is the mean transverse velocity, and v is the velocity fluctuation in the transverse direction. The kinematic Reynolds shear stress is denoted as $R_{uv} = -\langle uv\rangle$, where angle brackets $\langle\,\rangle$ denote Reynolds averaging. Note that the pressure gradient term and $\partial(\langle uu\rangle - \langle vv\rangle)/\partial x$ term are neglected in the mean momentum equation, as they can be shown to be insignificant for this flow.[1,13] The corresponding boundary conditions for planar turbulent wakes are listed in Table I.

In this paper, we apply a scaling patch analysis to investigate planar turbulent wakes. Scaling patch approach was originally developed to explore the multi-layer scaling and structure of wall-bounded turbulent flows,[32–35] and has been successfully applied to buoyancy-driven convection,[36] and recently to planar turbulent jet[37] and turbulent plumes.[38] Whereas some of the concepts and ideas in the scaling patch approach are similar to previous scaling approaches, the logical train of thought in the new approach is distinctly different.[39,40]

The objective of a scaling patch analysis is to reveal naturally the relative magnitudes of different terms in an engineering equation. Such an equation typically consists of the balance of more than two terms. However, different terms do not contribute equally to the balance of the equation. The relative magnitudes of terms are not clear when the equation is presented in a dimensional form. Through a systematic transformation of the dimensional equation into a dimensionless form, the scaling patch approach is able to determine the proper scales for the different terms in the equation. A key concept in the scaling patch approach is the admissible scaling, which requires that the scaled governing equations have at least two terms with a nominal order of magnitude 1, and the scaled boundary conditions should also be zero or nominal order of magnitude 1.[39,40]

The first step in a scaling patch approach is to normalize the flow variables by proper reference scales

$$U^* \stackrel{def}{=} \frac{U_\infty - U(x, y)}{U_{ref}(x)}, \tag{2}$$

TABLE I. Boundary conditions for planar turbulent wakes.

$y = 0$	$U = U_{ctr}$,	$V = 0$,	$R_{uv} = 0$
$y = \pm\infty$	$U = U_\infty$,	$V = 0$,	$R_{uv} = 0$

$$V^* \stackrel{def}{=} \frac{V(x,y)}{V_{\mathrm{ref}}(x)}, \tag{3}$$

$$R^*_{uv} \stackrel{def}{=} \frac{R_{uv}(x,y)}{R_{uv,\mathrm{ref}}(x)}. \tag{4}$$

The reference scales U_{ref}, V_{ref}, and $R_{uv,\mathrm{ref}}$ will be determined in the process of seeking an admissible scaling for the boundary conditions and the mean continuity and momentum equations. Far away from the wake generator, it has been observed that a proper length scale for the variation in the transverse direction is the wake width, and the normalized transverse distance is denoted as

$$\eta \stackrel{def}{=} \frac{y}{\delta(x)}, \tag{5}$$

where δ is a measure of the wake width. To transform the governing equations into the similarity variables, we first note that the derivatives of η with respect to x and y are

$$\frac{\partial\eta}{\partial x} = -\frac{1}{\delta}\frac{d\delta}{dx}\eta, \tag{6a}$$

$$\frac{\partial\eta}{\partial y} = \frac{1}{\delta}. \tag{6b}$$

Subsequently, the derivatives of U with respect to x is

$$\frac{\partial U}{\partial x} = -\frac{dU_{\mathrm{ref}}}{dx}U^* + \frac{U_{\mathrm{ref}}}{\delta}\frac{d\delta}{dx}\eta\frac{dU^*}{d\eta}, \tag{7}$$

and the derivatives of U, V, and R_{uv} with respect to y are

$$\frac{\partial U}{\partial y} = -\frac{U_{\mathrm{ref}}}{\delta}\frac{dU^*}{d\eta}, \tag{8a}$$

$$\frac{\partial V}{\partial y} = \frac{V_{\mathrm{ref}}}{\delta}\frac{dV^*}{d\eta}, \tag{8b}$$

$$\frac{\partial R_{uv}}{\partial y} = \frac{R_{uv,\mathrm{ref}}}{\delta}\frac{dR^*_{uv}}{d\eta}. \tag{8c}$$

Substituting the self-similar variables and their derivatives, the mean continuity equation and the mean momentum equation become

$$0 = -\frac{dU_{\mathrm{ref}}}{dx}U^* + \frac{U_{\mathrm{ref}}}{\delta}\frac{d\delta}{dx}\eta\frac{dU^*}{d\eta} + \frac{V_{\mathrm{ref}}}{\delta}\frac{dV^*}{d\eta}, \tag{9a}$$

$$0 = (U_\infty - U_{\mathrm{ref}}U^*)\frac{V_{\mathrm{ref}}}{\delta}\frac{dV^*}{d\eta} + (V_{\mathrm{ref}}V^*)\frac{U_{\mathrm{ref}}}{\delta}\frac{dU^*}{d\eta} - \frac{\nu U_{\mathrm{ref}}}{\delta^2}\frac{d^2U^*}{d\eta^2} + \frac{R_{uv,\mathrm{ref}}}{\delta}\frac{dR^*_{uv}}{d\eta}. \tag{9b}$$

The scaled boundary conditions are listed in Table II.

Based on the scaled boundary conditions in Table II, a natural scale for the mean axial velocity deficit is $U_{\mathrm{ref}} = U_\infty - U_{\mathrm{ctr}}$, because

TABLE II. Scaled boundary conditions for planar turbulent wakes.

$\eta = 0$	$U^* = \frac{U_\infty - U_{\mathrm{ctr}}}{U_{\mathrm{ref}}}$,	$V^* = 0$,	$R^*_{uv} = 0$.
$\eta = \pm\infty$	$U^* = 0$,	$V^* = 0$,	$R^*_{uv} = 0$.

the rescaled boundary conditions at $\eta = 0$ will be $U^*|_{\eta=0} = 1$, and all the other rescaled boundary conditions are 0. Next, proper scaling for V_{ref} and $R_{uv,\mathrm{ref}}$ is determined by seeking admissible scaling for the mean continuity equation and the mean momentum equation.

A. Admissible scaling of the continuity equation

Multiplying δ/V_{ref} onto Eq. (9a) produces a dimensionless continuity equation as

$$0 = -\left[\frac{\delta}{V_{\mathrm{ref}}}\frac{dU_{\mathrm{ref}}}{dx}\right]U^* + \left[\frac{U_{\mathrm{ref}}}{V_{\mathrm{ref}}}\frac{d\delta}{dx}\right]\eta\frac{dU^*}{d\eta} + \frac{dV^*}{d\eta}. \tag{10}$$

Integrating Eq. (10) from $\eta = 0$ (wake centerline) to $\eta = \infty$ and applying boundary conditions yields

$$0 = \left(\frac{\delta}{V_{\mathrm{ref}}}\frac{dU_{\mathrm{ref}}}{dx} + \frac{U_{\mathrm{ref}}}{V_{\mathrm{ref}}}\frac{d\delta}{dx}\right)\int_0^\infty U^* d\eta. \tag{11}$$

As $\int_0^\infty U^* d\eta$ is not zero, the integral constraint for the mean continuity equation is then

$$\frac{\delta}{V_{\mathrm{ref}}}\frac{dU_{\mathrm{ref}}}{dx} + \frac{U_{\mathrm{ref}}}{V_{\mathrm{ref}}}\frac{d\delta}{dx} = 0 \quad \text{or} \quad \frac{d\delta}{dx} = -\frac{\delta}{U_{\mathrm{ref}}}\frac{dU_{\mathrm{ref}}}{dx}. \tag{12}$$

This integral constraint has been derived and used by previous researchers, for example, Narasimha and Prabhu,[41] Moser *et al.*,[42] Liu,[19] and George and Davidson.[20] Integrating the integral constraint Eq. (12) yields

$$U_{\mathrm{ref}}(x)\delta(x) = \mathrm{const}. \tag{13}$$

Hence, to approach a self-similar state in the far wake, the product of $U_{\mathrm{ref}}(x)\delta(x)$ has to remain a constant. However, the integral constraint does not dictate individually any specific functional form for $\delta(x)$ or $U_{\mathrm{ref}}(x)$.

Applying the integral result Eq. (12), the continuity Eq. (10) can be rewritten as

$$0 = \left[\frac{U_{\mathrm{ref}}}{V_{\mathrm{ref}}}\frac{d\delta}{dx}\right]\left(U^* + \eta\frac{dU^*}{d\eta}\right) + \frac{dV^*}{d\eta}. \tag{14}$$

For Eq. (14) to be an admissible scaling, a natural scale for the mean transverse flow is

$$V_{\mathrm{ref}} = \frac{d\delta}{dx}U_{\mathrm{ref}} \quad \text{or} \quad V_{\mathrm{ref}} = -\delta\frac{dU_{\mathrm{ref}}}{dx}, \tag{15}$$

and the dimensionless continuity equation becomes

$$0 = \left(U^* + \eta\frac{dU^*}{d\eta}\right) + \frac{dV^*}{d\eta}. \tag{16}$$

Integrating Eq. (16) and applying boundary conditions produces a solution for the mean transverse flow as

$$V^* = -\eta\, U^*. \tag{17}$$

Equation (17) indicates that if U^* is self-similar (function of η only), V^* will be self-similar as well. Furthermore, as U^* is bounded between 0 and 1, V^* also varies between 0 and $\pm O(1)$ (η can be both positive and negative), a good indication of natural scaling in the scaling patch approach.[32,39,40]

Next, the analysis is applied to the mean axial momentum equation to determine a proper scaling for R_{uv} and provide an analytical solution for the scaled Reynolds shear stress.

B. Admissible scaling of the mean axial momentum equation

Multiplying $\delta/(U_\infty V_{\rm ref})$ onto Eq. (9b) produces a dimensionless mean momentum equation as

$$0 = \frac{dV^*}{d\eta} - \left[\frac{U_{\rm ref}}{U_\infty}\right]\left(U^*\frac{dV^*}{d\eta} - V^*\frac{dU^*}{d\eta}\right) - \left[\frac{U_{\rm ref}}{U_\infty}\right]\frac{\nu}{\delta V_{\rm ref}}\frac{d^2U^*}{d\eta^2} + \left[\frac{R_{uv,{\rm ref}}}{U_\infty V_{\rm ref}}\right]\frac{dR^*_{uv}}{d\eta}. \tag{18}$$

At a sufficient downstream distance in high Reynolds number turbulent wake flows, the centerline velocity deficit is much smaller than the free stream velocity, that is, $U_{\rm ref} = (U_\infty - U_{\rm ctr}) \ll U_\infty$. As a result, the second term on the right of Eq. (18) is a high-order term and can be neglected. The viscous term is also a high-order term due to the small pre-factor of $U_{\rm ref}/U_\infty$ and $\nu/(\delta V_{\rm ref})$. Thus, for Eq. (18) to be an admissible scaling, a proper scale for the kinematic Reynolds shear stress is

$$R_{uv,{\rm ref}} = U_\infty V_{\rm ref} = \frac{d\delta}{dx}U_\infty U_{\rm ref}, \tag{19}$$

and the mean momentum equation in the axial direction can be simplified as

$$0 = \frac{dV^*}{d\eta} + \frac{dR^*_{uv}}{d\eta}. \tag{20}$$

Integrating Eq. (20) and applying boundary conditions produces a solution to the Reynolds shear stress as

$$R^*_{uv} = -V^* = \eta U^*. \tag{21}$$

Hence, the shapes of the scaled Reynolds shear stress and mean transverse flow are the same, but with the opposite sign. The scaled Reynolds shear stress R^*_{uv} is also self-similar and varies within the range between 0 and $\pm O(1)$.

The scale of $(d\delta/dx)U_\infty U_{\rm ref}$ had been used in the past to scale the Reynolds shear stress,[18,20,21,42–44] but the direct connection between the $R_{uv,{\rm ref}}$ and $V_{\rm ref}$ had not been established. Here, we show explicitly that the proper scale for the Reynolds shear stress is a mixed scale of U_∞ and $V_{\rm ref}$.

III. COMPARISONS WITH EXPERIMENTAL DATA

In this section, the analytical results are compared with experimental data. Figure 2 presents the experimental measurements of the mean axial velocity deficit from several independent studies. The normalized mean axial velocity deficit profiles shown in the figure can be approximated by a simple Gaussian function[4,13]

$$U^* \approx e^{-a\eta^2}, \tag{22}$$

where $a = \ln(2) \approx 0.693$ arising from the definition of wake half-width δ as the location where $U^*(\eta = 1) = 0.5$. To better fit the experimental data, in particular near the wake edge, Wygnanski *et al.*[6] and Liu *et al.*[16] had proposed a slightly more complicated function

$$U^*(\eta) \approx e^{(-0.637\eta^2 - 0.056\eta^4)}. \tag{23}$$

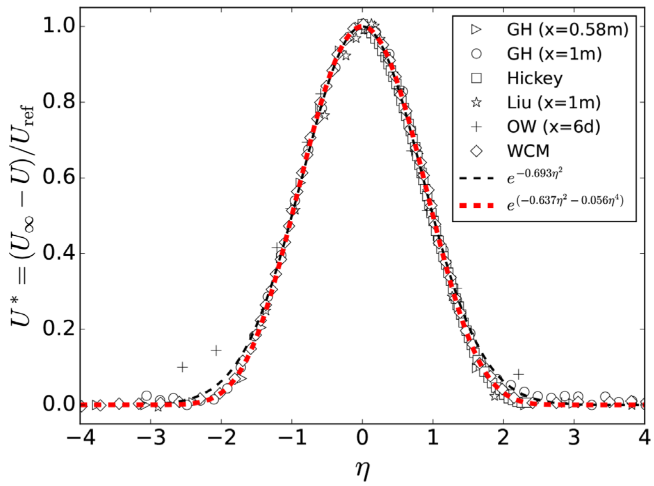

FIG. 2. Mean axial velocity deficit and approximate functions of Eqs. (22) and (23). Experimental data are from Gough and Hancock[11] (GH), Hickey,[28] Liu,[15] Ong and Wallace[12] (OW), and Wygnanski *et al.*[6] (WCM).

A. Mean transverse velocity profiles

In experimental studies of wake flows, the mean transverse velocity measurement is rarely presented. Gough and Hancock[11] measured the mean transverse velocity in a relatively low Reynolds number wake flow. As shown in Fig. 3(a), the mean transverse velocity profile of a wake flow is anti-symmetric about the wake centerline, indicating that ambient fluids move toward the wake core from the bottom and top. From Eq. (17), the mean transverse velocity can be approximated as

$$V^*(\eta) \approx -\eta\, e^{-0.693\eta^2}, \tag{24a}$$

$$V^*(\eta) \approx -\eta\, e^{(-0.637\eta^2 - 0.056\eta^4)}. \tag{24b}$$

The approximation equations for V^* exhibit a peak and a trough within the wake. The locations of the peak and trough can be found by solving $dV^*/d\eta = 0$. The peak transverse velocity location and value are

$$\eta \approx \pm 1/\sqrt{2\ln(2)} \approx \pm 0.85, \tag{25a}$$

$$|V^*|_{\max} \approx 0.515. \tag{25b}$$

Figure 3(a) shows that the approximate Eq. (24a) or Eq. (24b) capture well the shape of the measured mean transverse velocity. The peak and trough locations in the experimental data shown in Fig. 3(a) agree well with the analytical approximation; there is scatter between the peak value and the analytical result of $|V^*|_{\max} = 0.515$. The scatter can be attributed to (1) uncertainty in the measurements of V and (2) uncertainty in the calculation of $V_{\rm ref} = U_{\rm ref}\, d\delta/dx$. There is uncertainty in the determination of the wake width δ, and uncertainty in the calculation of $d\delta/dx$ using simple finite difference.

The analytical result for V is only valid in the far wake for $(U_\infty - U_{\rm ctr})/U_\infty \ll 1$, as "self-similarity" assumption is used to obtain the continuity equation in the form of Eq. (9a) or Eq. (10). Figure 3(b) shows that $|V|_{\max}/V_{\rm ref}$ asymptotically approaches to the analytical result of 0.515 when $(U_\infty - U_{\rm ctr})/U_\infty$ becomes smaller than 0.2, which occurs in the far field of the wake. As shown in Liu

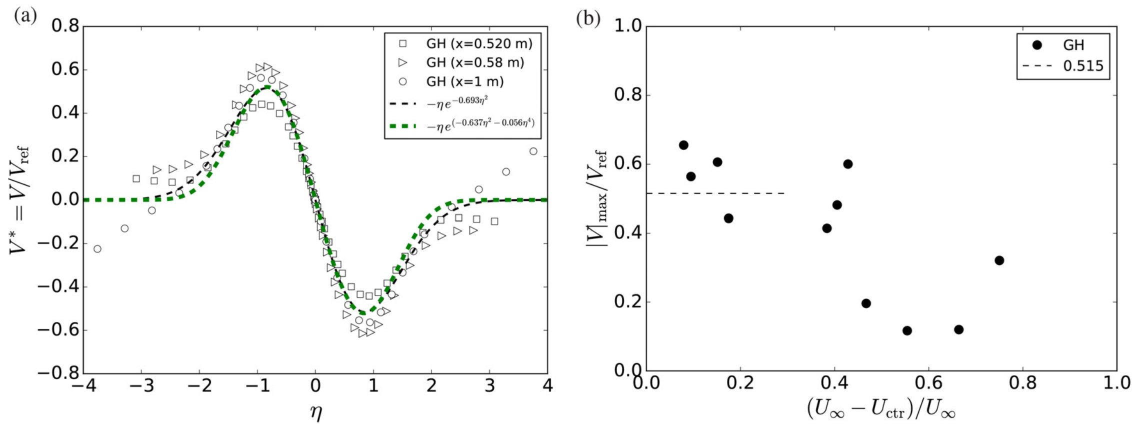

FIG. 3. (a) Mean transverse velocity and approximate functions of Eqs. (24a) and (24b). (b) Maximum mean transverse velocity normalized by $V_{\mathrm{ref}} = U_{\mathrm{ref}} d\delta/dx$ vs $(U_\infty - U_{\mathrm{ctr}})/U_\infty$. Data of Gough and Hancock[11] (GH).

et al.,[16] when scaled by $\delta_{0.5}$ and $U_\infty - U_{\mathrm{ctr}}$, the mean velocity defect profiles in planar turbulent wakes under zero pressure gradient, adverse pressure gradient, or favorable pressure gradient all collapse onto a "universal" shape for $x/\theta_0 > 40$. Here, θ_0 is the initial wake momentum thickness. Thus $x/\theta_0 \approx 40$ can be viewed as the bound of the far wake. For $(U_\infty - U_{\mathrm{ctr}})/U_\infty > 0.2$, Fig. 3(b) shows that $|V|_{\max}/V_{\mathrm{ref}}$ is smaller than 0.515, indicating that the reference velocity scale valid in the self-similar region $V_{\mathrm{ref}} = U_{\mathrm{ref}} d\delta/dx$ overestimates the mean transverse velocity flow in the near-wake region. The scatter in Fig. 3(b) is mainly attributed to the uncertainties mentioned above.

B. Reynolds shear stress profiles

Using a Gaussian function for U^*, the analytical Eq. (21) for the scaled Reynolds shear stress can be approximated as

$$R^*_{uv}(\eta) \approx \eta\, e^{-0.693\eta^2}, \tag{26a}$$

$$R^*_{uv}(\eta) \approx \eta\, e^{(-0.637\eta^2 - 0.056\eta^4)}. \tag{26b}$$

The approximation functions for R^*_{uv} also exhibit a peak and a trough within the wake. The location and value of the peak and trough of the scaled Reynolds shear stress profile are also $\eta = \pm 0.85$ and $|R^*_{uv}|_{\max} \approx 0.515$.

Figure 4(a) shows that the approximate equations capture the experimental measurements of the Reynolds shear stress reasonably well. The peak and trough locations of the Reynolds shear stress profiles agree well with the analytical result, but there is scattering in the peak and trough values.

Figure 4(b) shows that, far from the wake generator or $(U_\infty - U_{\mathrm{ctr}})/U_\infty \lesssim 0.2$, $U_\infty V_{\mathrm{ref}}$ is a proper scale for the Reynolds shear stress as its maximum value scaled by this reference scale asymptotically approach a constant around 0.515, which is of a nominal order 1 as required by the admissible scaling. For $(U_\infty - U_{\mathrm{ctr}})/U_\infty > 0.2$, $|R_{uv}|_{\max}/(U_\infty V_{\mathrm{ref}})$ is smaller than 0.515, indicating that $U_\infty V_{\mathrm{ref}}$ overpredicts the Reynolds shear stress in the near-wake region and is not an appropriate scale when the wake flow has not reached a self-similar state.

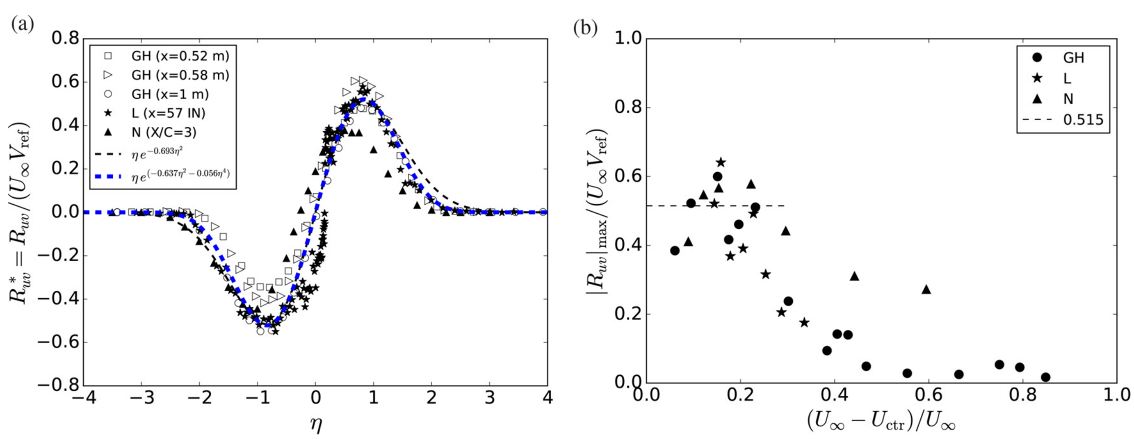

FIG. 4. (a) Reynolds shear stress and approximate functions from Eqs. (26a) and (26b). (b) Maximum Reynolds shear stress normalized by $U_\infty V_{\mathrm{ref}}$ vs $(U_\infty - U_{\mathrm{ctr}})/U_\infty$. Data of Gough and Hancock[11] (GH), data of Liu[15] (L), and data of Nakayama[30] (N).

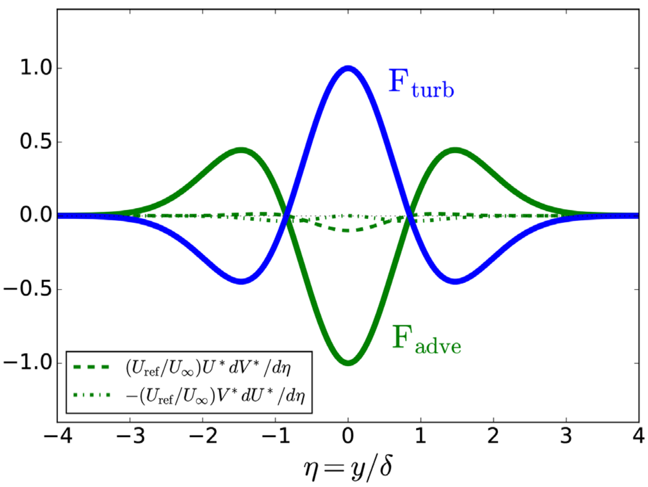

FIG. 5. Distribution of terms in the mean momentum equation. Turbulence force (per unit volume) is the Reynolds shear stress gradient $F_{\mathrm{turb}} = dR^*_{uv}/d\eta$. Advective force is $F_{\mathrm{adve}} = dV^*/d\eta + (U_\infty - U_{\mathrm{ctr}})/U_\infty\,(-U^*dV^*/d\eta + V^*dU^*/d\eta)$.

C. Characteristic of force balance in planar turbulent wakes

Using the approximate functions for U^*, V^*, and R^*_{uv} in the self-similar regions, the terms in the mean momentum Eq. (18) are calculated and plotted in Fig. 5. Far away from the wake generator, the momentum equation is balanced by two forces: advective force and turbulent force (gradient of Reynolds shear stress), while the viscous force or the term of $\partial R_{uu}/\partial x$ is negligible. Near the wake core, the Reynolds shear stress gradient is positive (a driving force) and the advective force is negative (a drag force). Away from the wake core, Reynolds shear stress gradient becomes negative (or a drag force), and the advective force becomes a driving force of the wake flow.

The advective force can be decomposed as $F_{\mathrm{adve}} = dV^*/d\eta + (U_\infty - U_{\mathrm{ctr}})/U_\infty\,(-U^*dV^*/d\eta + V^*dU^*/d\eta)$. The dominant contribution to the advective force is the term of $dV^*/d\eta$, because the other terms have a pre-factor of $(U_\infty - U_{\mathrm{ctr}})/U_\infty$, which is small in the far wake. In Fig. 5, $(U_\infty - U_{\mathrm{ctr}})/U_\infty$ is set as 0.1 for the illustration purpose. At smaller $(U_\infty - U_{\mathrm{ctr}})/U_\infty$ in the far wake, the contribution of $(U_\infty - U_{\mathrm{ctr}})/U_\infty\,(-U^*dV^*/d\eta + V^*dU^*/d\eta)$ is smaller than that in the figure. In the near-wake region where $U_\infty - U_{\mathrm{ctr}}$ is not much smaller than U_∞, the flow has not reached a self-similar state and the contributions of different terms in the mean momentum equation are more complicated than that sketched in Fig. 5. Moreover, the relative contributions of different forces vary in the axial direction in the near-wake region.

IV. CONCLUSIONS

A relatively new scaling patch approach is used in this work to investigate the scaling properties of planar turbulent wakes. By seeking an admissible scaling for the mean continuity equation, a proper scale for the mean transverse flow is found as $V_{\mathrm{ref}} = (d\delta/dx)U_{\mathrm{ref}}$. From an admissible scaling of the mean momentum equation, a proper scale for the kinematic Reynolds shear stress is found as $R_{uv,\mathrm{ref}} = U_\infty V_{\mathrm{ref}}$, which is a mixed scale of U_∞ and V_{ref}. In the far wake region, an analytical equation is derived for the scaled mean transverse flow as $V^* = -\eta U^*$, and analytical equation for the scaled Reynolds shear stress as $R^*_{uv} = \eta U^*$. Using a Gaussian function for U^*, approximate functions for V^* and R^*_{uv} are presented and are found to agree well with experimental data. The main results of this work are summarized in Table III.

TABLE III. Summary of planar turbulent wakes far from the wake generator ($U_\infty - U_{\mathrm{ctr}} \ll U_\infty$).

Mean axial velocity scale	$U_{\mathrm{ref}} = U_\infty - U_{\mathrm{ctr}}$
Mean transverse velocity scale	$V_{\mathrm{ref}} = U_{\mathrm{ref}}\dfrac{d\delta}{dx}$
Reynolds shear stress scale	$R_{uv,\mathrm{ref}} = U_\infty V_{\mathrm{ref}}$
Admissible scaling of continuity equation	$0 = -\left(U^* + \eta\dfrac{dU^*}{d\eta}\right) + \dfrac{dV^*}{d\eta}$
Admissible scaling of momentum equation	$0 = \dfrac{dV^*}{d\eta} + \dfrac{dR^*_{uv}}{d\eta}$
Analytical equation for V^*	$V^* = -\eta U^* \approx -\eta e^{-0.693\eta^2}$
Analytical equation for R^*_{uv}	$R^*_{uv} = \eta U^* \approx \eta e^{-0.693\eta^2}$

Proper scaling of turbulent flow variables is important in presenting and evaluating experimental/numerical data. More importantly, proper scaling is essential in understanding the underlying physics and developing turbulence models. In this work, we reveal that proper scaling lies at the root of a fundamental question in turbulent wake flows whether a universal self-similar state exists in the fields far from its initiation. By applying proper scalings to the flow variables, the mean axial flow, the mean transverse flow, and the Reynolds shear stress do approach a self-similar state in the far field of planar turbulent wakes.

Although some of the relations have been found in the past, this work uses a systematic approach to derive the scaling properties in planar turbulent wakes. One key finding is the important role of the mean transverse flow in the understanding and scaling of turbulent wake flows.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Tie Wei: Conceptualization (equal); Data curation (equal); Formal analysis (equal); Investigation (equal); Writing – original draft (equal). **Daniel Livescu:** Conceptualization (equal); Funding acquisition (equal); Writing – review & editing (equal). **Xiaofeng Liu:** Data curation (equal); Formal analysis (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created in this study.

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