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Tie Wei[1,a)] and Daniel Livescu[2,b)]

AFFILIATIONS

[1]Department of Mechanical Engineering, New Mexico Institute of Mining and Technology, Socorro, New Mexico 87801, USA
[2]CCS-2, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA

[a)]**Author to whom correspondence should be addressed:** tie.wei@nmt.edu
[b)]**Electronic mail:** livescu@lanl.gov

ABSTRACT

Proper scaling in turbulent planar plumes is investigated here using a scaling patch approach. Based on the scaled boundary conditions, a proper velocity scale for the mean axial flow is the plume centerline velocity $U_{\mathrm{ref}} = U_{\mathrm{ctr}}$, and a proper temperature scale for the temperature excess is $\Theta_{\mathrm{ref}} = T_{\mathrm{ctr}} - T_{\infty}$, where T_{ctr} is the plume centerline temperature and T_{∞} is the ambient fluid temperature. By seeking an admissible scaling, a key concept in the scaling patch approach, for the mean continuity, mean momentum, and mean energy equations, respectively, the following is found: (1) a proper scale for the mean transverse flow is $V_{\mathrm{ref}} = (d\delta/dx)U_{\mathrm{ctr}}$, where $d\delta/dx$ is the growth rate of the plume width. (2) A proper scale for the Reynolds shear stress is $R_{vu,\mathrm{ref}} = U_{\mathrm{ctr}}V_{\mathrm{ref}} = (d\delta/dx)U_{\mathrm{ctr}}^2$, a mix of the scales for the mean axial and transverse flows. (3) A proper scale for the turbulent heat flux is $R_{v\theta,\mathrm{ref}} = V_{\mathrm{ref}}\Theta_{\mathrm{ctr}}$, a mix of the scales for the mean transverse flow and mean temperature excess. The mean transverse flow thus plays a critical role in the scaling of turbulent planar plumes. Approximate functions are developed for the scaled mean transverse flow, Reynolds shear stress, and turbulent temperature flux, and are found to agree favorably with experimental and numerical simulation data. The integral analysis of the mean momentum equation yields a Richardson number Ri, which remains invariant in the axial direction. The Richardson number is defined as $\mathrm{Ri} \stackrel{\mathrm{def}}{=} g\beta\Theta_{\mathrm{ctr}}\delta_t/(U_{\mathrm{ctr}}V_{\mathrm{ref}}) \approx 1/\sqrt{2}$, where g is the gravitational acceleration, β is the thermal expansion coefficient, and δ_t is the plume half-width based on the mean temperature profile. This Richardson number arises directly from the scaling patch analysis of the mean momentum equation, including both the streamwise and transverse velocity scales.

I. INTRODUCTION

Plumes are generated from a source of buoyancy, for example, smoke rising from a burning cigarette. The fluid in contact with the heat source reaches a higher temperature. The temperature excess, in turn, creates a local density deficiency relative to the ambient fluid, driving lighter fluids upwards in a gravitational field. As the lighter fluids rise in the vertical direction, ambient fluids are entrained to broaden the plume. An image of a planar plume is illustrated in Fig. 1. The axial or vertical direction is in the x-direction, and the transverse flow is in the y-direction.[1]

Plumes are ubiquitous in both natural and man-made environments, for example, fire plumes, cooling tower plumes, and chimney exhausts. Fire plumes have been studied extensively to improve the design of smoke detectors and sprinkler systems in buildings and atriums of large shopping plazas. In many coastal cities, pretreated sewage is discharged as buoyant plumes through submarine outfalls located on the sea bed, and a better understanding of plumes can be used to improve water quality control and for risk assessment. On the theoretical side, knowledge of buoyant plumes advances our understanding of general free shear turbulence and, in particular, the buoyancy effect on the structure of free shear turbulence. Therefore, turbulent plumes have been investigated by numerous researchers, experimentally, analytically, and numerically.[2–14]

While the scaling for the mean axial flow (in the streamwise or vertical direction in Fig. 1) and mean temperature distributions in turbulent plumes is well established,[15,16] the proper scaling for the mean transverse flow, the Reynolds shear stress, and turbulent heat flux has not been settled. The magnitude of the mean transverse flow in turbulent plumes is very small, lower than 0.01 m/s in most experimental studies. Therefore, it is extremely challenging to obtain accurate mean transverse velocity measurements. Traditional analyses of turbulent plumes avoid the explicit scaling of V by integrating the mean continuity equation. The purpose of the present paper is to determine explicitly the proper scaling of the mean transverse flow, the Reynolds shear stress, and turbulent heat flux in turbulent planar plumes using a relatively new scaling patch approach, and clarify the relations among the

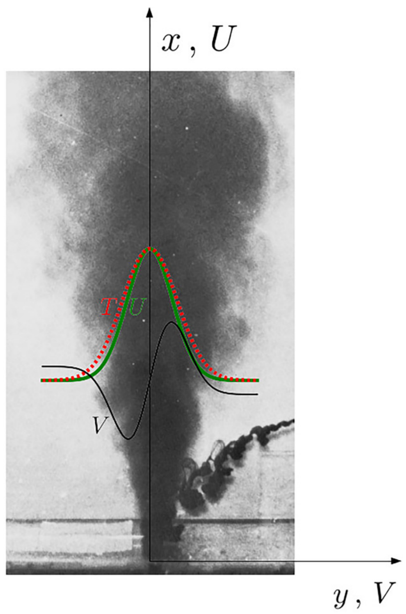

FIG. 1. Illustration of a planar plume. Image is from the experiment of Kotsovinos.[1] *x* denotes the axial direction pointing upwards, and *y* denotes the transverse direction. Gravity is in the vertical direction, pointing downwards. The shapes of the mean axial velocity *U*, mean transverse velocity *V*, and the mean temperature *T* are also sketched.

scalings in the mean axial and transverse flows and turbulent transport terms.

One of the earliest theoretical analyses of plumes dates back to 1937, by Zeldovich,[17] who performed a similarity analysis of turbulent plumes. The first quantitative plume study was carried out by Schmidt in 1941,[18] who used mixing-length hypotheses to obtain expressions for the mean velocity and temperature profiles in both planar and round plumes. In the 1950s, more similarity analyses were conducted by Rouse *et al.*,[19] Batchelor,[20] Morton *et al.*,[21] and Priestley and Ball.[22] Rouse *et al.*[19] derived similarity solutions for planar and round plumes and verified the solutions with experimental measurements in plumes above a single gas burner and above a line of gas burners. Batchelor[20] proposed similarity solutions for turbulent plumes for both planar and round geometries in neutral and stratified environments. Morton *et al.*[21] made three assumptions about plumes including (i) the profiles of the vertical velocity and buoyancy are similar at all heights, (ii) the entrainment rate at any height is proportional to the characteristic velocity at that height, and (iii) the fluids are incompressible and do not change the volume on mixing. Priestley and Ball[22] did not make an assumption about the entrainment, but assumed that the mean shear stress is proportional to the square of the mean plume velocity. Morton *et al.*[21] extended the analysis to nonsimilar situations. The classical plume theory developed by Zeldovich, Rouse *et al.*, Batchelor, and Morton *et al.* was reviewed by Hunt and Van den Bremer.[23] More reviews on plumes were given by Chen and Rodi,[24] List,[25] Ramaprian and Chandrasekhara,[15] and Baines.[26]

Broadly speaking, previous analytical studies of plumes can be classified into three categories: dimensional, integral, and differential analyses. Knowledge gained from dimensional analysis is insightful but limited because the governing equations are not directly employed. Integral analysis examines the flow through integration of the mass, momentum, and energy equations; to account for the turbulent effects, certain assumptions have to be made. One popular assumption is the universal entrainment coefficient.[21] However, experimental measurements have indicated that the assumption of a universal entrainment coefficient is incorrect.[16] In previous differential analyses of plumes, the mean transverse flow was not explicitly addressed. Instead, the mean continuity equation was integrated to avoid the explicit analysis of the mean transverse flow.

High-quality measurements are critical to appraise the analyses of turbulent flows, including turbulent planar plumes. In the 1970s and 1980s, two comprehensive experimental studies of turbulent planar plumes were performed by Kotsovinos[16] and Ramaprian and Chandrasekhara.[15] Recently, numerical simulations have become an important tool in investigating turbulent planar plumes. Numerical simulations of planar plumes have been performed by Malin and Spalding,[27] Kalita *et al.*,[7] and Dewan *et al.*[9]

In this paper, a relatively new scaling patch approach is applied to determine the proper scaling in turbulent planar plumes. Scaling patch approach was originally developed for shear-driven wall-bounded turbulence by Fife and coworkers.[28–31] Whereas some of the concepts and ideas in the scaling patch approach are similar to previous scaling approaches, the logical train of thought in the new approach is distinctly different.[30,31] The scaling patch approach has been applied to passive scalar transport in the turbulent pipe or channel flow,[32–34] turbulent boundary flow with roughness,[35,36] turbulent Taylor–Couette flow,[37] buoyancy-driven turbulent convection,[38,39] and more recently in turbulent planar jet.[40]

The objective of scaling patch analysis is to reveal naturally the relative magnitudes of different terms in an engineering equation. Such an equation typically consists of the balance of more than two terms. However, different terms do not contribute equally to the balance of the equation. The relative magnitudes of terms are not clear when the equation is presented in a dimensional form. Through a systematic transformation of the dimensional equation into a dimensionless form, the scaling patch approach is able to determine the proper scale for each term in the equation. A balance equation can be scaled in any number of ways, creating an infinite number of versions of dimensionless equations, and all versions are mathematically equivalent; that is, one scaling can be transformed to another by simple rescaling factors.[30,31] However, only a certain scaling reflects naturally the local balance of terms. For example, in a region of the flow, if the scaled distance is of $O(1)$, then the leading order balance within that region should also be of $O(1)$.

In scaling patch analysis of turbulent flow, the governing equations for different moments are transformed into dimensionless forms, as an admissible scaling. For a dimensionless equation to be an admissible scaling, at least two terms must have a nominal order of magnitude 1.[30,31,39] Here, we show that the admissible scaling can clearly reveal the relative magnitude of terms in the mean governing equations of turbulent planar plume, which in turn assists in determining the proper scaling of the mean transverse flow, Reynolds shear stress, and turbulent heat flux.

In Sec. II, the governing equations for the mean flow and heat transport are presented. In Sec. III, proper scaling in the far field of turbulent plumes is determined by seeking admissible scaling for the mean continuity equation, the mean energy equation, and the mean momentum equation in the axial direction. Approximate functions for the scaled mean transverse flow, kinematic Reynolds shear stress, and turbulent temperature flux are developed and compared with experimental data. Section IV summarizes the work.

II GOVERNING EQUATIONS OF TURBULENT PLUMES

As in other free-shear turbulent flows, planar plumes are "slender"; that is, they spread slowly in the transverse direction. Therefore, Prandtl's boundary layer equations are used to describe turbulent planar plumes.[41,42] In this work, we focus on a region far away from the plume origin and assume a statistical steady state. The mean continuity, mean momentum equation, and mean energy equations, respectively, are

$$0 = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}, \tag{1a}$$

$$0 = -\left\{ U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} \right\} + \frac{\partial R_{vu}}{\partial y} + g\beta\Theta, \tag{1b}$$

$$0 = -\left\{ U\frac{\partial \Theta}{\partial x} + V\frac{\partial \Theta}{\partial y} \right\} + \frac{\partial R_{v\theta}}{\partial y}, \tag{1c}$$

where g is the gravitational acceleration pointing in the negative x-direction, and β is the thermal expansion coefficient. Here, an uppercase letter denotes a mean flow or temperature variable. U is the mean axial velocity and V is the mean transverse velocity, T is the mean temperature, T_∞ is the ambient temperature, and $\Theta \overset{\text{def}}{=} T - T_\infty$ is the mean temperature excess. $R_{vu} = -\langle vu \rangle$ is the kinematic Reynolds shear stress, and $R_{v\theta} = -\langle v\theta \rangle$ is the turbulent temperature flux. A lowercase letter denotes a fluctuation quantity. u and v are the velocity fluctuations in the streamwise and transverse directions, respectively, and θ is the temperature fluctuation. The angle brackets $\langle\,\rangle$ denote Reynolds averaging.

In the mean momentum Eq. (1b), the Oberbeck–Boussinesq approximation is used to approximate the buoyancy force by the temperature excess,[41,42] as $g\beta(T - T_\infty)$ or $g\beta\Theta$. For most fluids of engineering interest, within a limited variation of temperature, β is nearly a constant. Note that it is assumed that the plume is fully turbulent and the viscous force and molecular heat diffusion are negligible compared to the turbulent transport of momentum or heat.[41]

The corresponding boundary conditions for turbulent planar plumes are listed in Table I. Due to symmetry, the Reynolds shear stress and turbulent temperature flux are zero at the plume centerline.

III. SCALING ANALYSIS OF THE GOVERNING EQUATIONS

It is observed that at a sufficient distance from the plume origin ($x \gtrsim 20D$ where D is the nozzle height[15]) the planar plume approaches a self-similar state; that is, the properly scaled mean flow profiles at different axial locations merge onto a single curve.[43] Here, the self-similar variables are defined as follows

TABLE I. Boundary conditions at the plume centerline $y = 0$ and far away from the plume $y = \pm\infty$.

$y = 0$	$U = U_{ctr},\ V = 0,\ \ \Theta = T_{ctr} - T_\infty,\ R_{vu} = 0,\ R_{v\theta} = 0.$
$y = \pm\infty$	$U = 0,\ \ V = V_{\pm\infty},\ \Theta = 0,\ \ R_{vu} = 0,\ R_{v\theta} = 0.$

$$\eta \overset{\text{def}}{=} \frac{y}{\delta(x)}, \tag{2a}$$

$$U^*(\eta) \overset{\text{def}}{=} \frac{U(x,y)}{U_{ref}(x)},\quad V^*(\eta) \overset{\text{def}}{=} \frac{V(x,y)}{V_{ref}(x)},\quad \Theta^*(\eta) \overset{\text{def}}{=} \frac{T(x,y) - T_\infty}{\Theta_{ref}(x)}, \tag{2b}$$

$$R^*_{vu}(\eta) \overset{\text{def}}{=} \frac{R_{vu}(x,y)}{R_{vu,ref}(x)},\quad R^*_{v\theta}(\eta) \overset{\text{def}}{=} \frac{R_{v\theta}(x,y)}{R_{v\theta,ref}(x)}, \tag{2c}$$

where δ is a measure of the plume width. $U_{ref}, V_{ref}, T_{ref}, R_{vu,ref}, R_{v\theta,ref}$, respectively, are the proper scales for the mean axial velocity, the mean transverse velocity, the temperature excess, the kinematic Reynolds shear stress, and the turbulent transport of heat. These scales will be determined in the following analysis.

To express the governing equations using the self-similar variables, we first note that the derivatives of η with respect to x and y are

$$\frac{\partial \eta}{\partial x} = -\frac{1}{\delta}\frac{d\delta}{dx}\eta, \tag{3a}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{\delta}. \tag{3b}$$

Subsequently, the derivatives of U and Θ with respect to x are

$$\frac{\partial U}{\partial x} = \frac{\partial(U_{ref}U^*)}{\partial x} = \frac{dU_{ref}}{dx}U^* + U_{ref}\frac{dU^*}{d\eta}\frac{\partial \eta}{\partial x} = \frac{dU_{ref}}{dx}U^* - \frac{U_{ref}}{\delta}\frac{d\delta}{dx}\eta\frac{dU^*}{d\eta}, \tag{4a}$$

$$\frac{\partial \Theta}{\partial x} = \frac{\partial(\Theta_{ref}\Theta^*)}{\partial x} = \frac{d\Theta_{ref}}{dx}\Theta^* + \Theta_{ref}\frac{d\Theta^*}{d\eta}\frac{\partial \eta}{\partial x} = \frac{d\Theta_{ref}}{dx}U^* - \frac{\Theta_{ref}}{\delta}\frac{d\delta}{dx}\eta\frac{d\Theta^*}{d\eta}, \tag{4b}$$

and the derivatives of U, Θ, V, R_{vu}, and $R_{v\theta}$ with respect to y are

$$\frac{\partial U}{\partial y} = \frac{U_{ref}}{\delta}\frac{dU^*}{d\eta}, \tag{5a}$$

$$\frac{\partial \Theta}{\partial y} = \frac{\Theta_{ref}}{\delta}\frac{d\Theta^*}{d\eta}, \tag{5b}$$

$$\frac{\partial V}{\partial y} = \frac{V_{ref}}{\delta}\frac{dV^*}{d\eta}, \tag{5c}$$

$$\frac{\partial R_{vu}}{\partial y} = \frac{R_{vu,ref}}{\delta}\frac{dR^*_{vu}}{d\eta}, \tag{5d}$$

$$\frac{\partial R_{v\theta}}{\partial y} = \frac{R_{v\theta,ref}}{\delta}\frac{dR^*_{v\theta}}{d\eta}. \tag{5e}$$

Substituting the definitions of the self-similar variables in Eqs. (2a), (2b), (2c) and their derivatives in Eqs. (4) and (5) into the mean continuity Eq. (1a), the mean momentum Eq. (1b), and the mean energy Eq. (1c), the mean equations can be expressed, using the self-similar variables, as

$$0 = \frac{dU_{ref}}{dx}U^* - \frac{U_{ref}}{\delta}\frac{d\delta}{dx}\eta\frac{dU^*}{d\eta} + \frac{V_{ref}}{\delta}\frac{dV^*}{d\eta}, \tag{6a}$$

$$0 = \frac{U_{\rm ref} V_{\rm ref}}{\delta} U^* \frac{dV^*}{d\eta} - \frac{U_{\rm ref} V_{\rm ref}}{\delta} V^* \frac{dU^*}{d\eta} + \frac{R_{\rm vu,ref}}{\delta} \frac{dR^*_{\rm vu}}{d\eta} + g\beta\Theta_{\rm ref}\Theta^*, \tag{6b}$$

$$0 = -U_{\rm ref} \frac{d\Theta_{\rm ref}}{dx} U^*\Theta^* + \frac{U_{\rm ref}\Theta_{\rm ref}}{\delta} \frac{d\delta}{dx} \eta U^* \frac{d\Theta^*}{d\eta} - \frac{V_{\rm ref}\Theta_{\rm ref}}{\delta} V^* \frac{d\Theta^*}{d\eta} + \frac{R_{\rm v\theta,ref}}{\delta}\frac{dR^*_{\rm v\theta}}{d\eta}. \tag{6c}$$

The scaled boundary conditions at the centerline and far away from the plume are listed in Table II.

In this paper, the proper scales for $U_{\rm ref}$, $V_{\rm ref}$, $R_{\rm vu,ref}$, and $R_{\rm v\theta,ref}$ are determined by the scaling patch approach. In this approach, the proper scaled boundary conditions should be either zero or of $O(1)$. As listed in Table II, all the scaled boundary conditions are 0, except for $U^*|_{\eta=0}$, $\Theta^*|_{\eta=0}$, and $V^*|_{\eta=\pm\infty}$. Hence, setting the scaled boundary conditions $U^*|_{\eta=0} = 1$ and $\Theta^*|_{\eta=0} = 1$ leads to the commonly used scales for the mean axial flow and mean temperature excess[41] as $U_{\rm ref} = U_{\rm ctr}$ and $\Theta_{\rm ref} = T_{\rm ctr} - T_\infty$, respectively. The proper scale for the mean transverse flow $V_{\rm ref}$ will be determined from the analysis of the mean continuity equation. It turns out that $|V_\infty| \approx -1.06 V_{\rm ref}$ [see Eqs. (13) and (16) below]. In other words, the scaled boundary condition for the mean transverse flow is also of order $|V^*|_{\eta=\infty} = O(1)$.

Next, we will first determine a proper scale for the mean transverse flow by seeking an admissible scaling for the mean continuity equation, then a proper scale for the turbulent heat flux from the mean energy equation, and a proper scale of the Reynolds shear stress from the mean momentum equation in the axial direction.

A. Admissible scaling of the mean continuity equation

To transform the mean continuity equation into a dimensionless form, Eq. (6a) is multiplied by $\delta/V_{\rm ref}$ to yield

$$0 = \left[\frac{\delta}{V_{\rm ref}} \frac{dU_{\rm ctr}}{dx}\right] U^* - \left[\frac{U_{\rm ctr}}{V_{\rm ref}} \frac{d\delta}{dx}\right] \eta \frac{dU^*}{d\eta} + \frac{dV^*}{d\eta}. \tag{7}$$

The nominal orders of magnitudes of the three terms on the right side of Eq. (7) are $\delta/V_{\rm ref}\, dU_{\rm ctr}/dx$, $U_{\rm ctr}/V_{\rm ref}\, d\delta/dx$, and 1. For Eq. (7) to be an admissible scaling, that is, at least two terms with a nominal order of magnitude 1, one choice is to set the scale for the mean transverse flow as

$$V_{\rm ref} = U_{\rm ctr} \frac{d\delta}{dx}, \tag{8}$$

and the dimensionless continuity equation becomes

$$0 = BU^* - \eta \frac{dU^*}{d\eta} + \frac{dV^*}{d\eta}, \tag{9}$$

where B denotes the ratio

TABLE II. Scaled boundary conditions at the plume centerline $\eta = 0$ and far away from the plume centerline $\eta = \pm\infty$.

$\eta = 0$	$U^* = \frac{U_{\rm ctr}}{U_{\rm ref}}$, $V^* = 0$, $\Theta^* = \frac{T_{\rm ctr} - T_\infty}{\Theta_{\rm ref}}$, $R^*_{\rm vu} = 0$, $R^*_{\rm v\theta} = 0$.
$\eta = \pm\infty$	$U^* = 0$, $V^* = \frac{V_{\pm\infty}}{V_{\rm ref}}$, $\Theta^* = 0$, $R^*_{\rm vu} = 0$, $R^*_{\rm v\theta} = 0$.

$$B \stackrel{\rm def}{=} \frac{\delta}{U_{\rm ctr}} \frac{\dfrac{dU_{\rm ctr}}{dx}}{\dfrac{d\delta}{dx}}. \tag{10}$$

Empirically, it is observed that $U_{\rm ctr}$ is approximately constant in the far fields of turbulent planar plumes,[15] which leads to $dU_{\rm ctr}/dx = 0$ and $B = 0$. Physically, along the centerline, the plume is under the opposing actions of buoyancy and turbulent shear forces (see Fig. 7). The constant centerline plume velocity implies a balance of the two opposing forces. As B is zero, the mean continuity Eq. (9) can be simplified as

$$0 = -\eta \frac{dU^*}{d\eta} + \frac{dV^*}{d\eta}. \tag{11}$$

An exact solution for the mean transverse velocity can be obtained by integrating Eq. (9) in the transverse direction from 0 to η as

$$V^* = \eta U^* - \int_0^\eta U^* d\eta. \tag{12}$$

The mean transverse velocity at plume edge is then

$$V^*_\infty = -\int_0^\infty U^* d\eta. \tag{13}$$

Note that V^*_∞ is negative and $V^*_{-\infty}$ is positive, since ambient fluids are entrained from two sides toward the core of the plume.

Empirically, it has been observed that the mean axial velocity and the mean temperature excess can be approximated by a Gaussian function[43] as

$$U^*(\eta) \approx e^{-a\eta^2}, \tag{14}$$

where a is a constant. If the plume half-width is defined as $U^*(y = y_{0.5u}) = 0.5$, then $a = \ln(2)$. The Gaussian approximation given by Eq. (14) is compared with experimental data in Fig. 2.

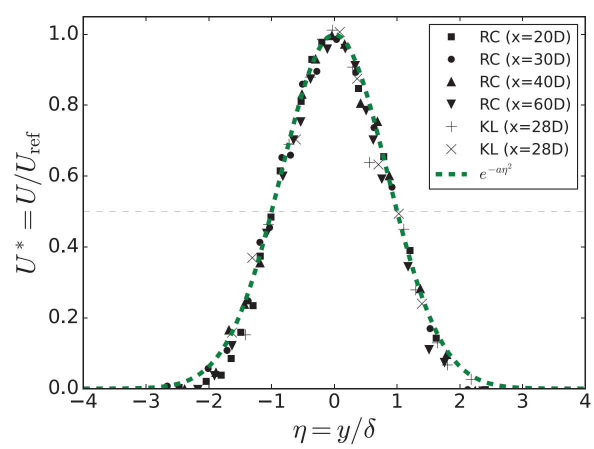

FIG. 2. Comparison of the scaled mean axial velocity U^* data with Eq. (14) (dashed green curve). The reference mean axial velocity is the plume centerline velocity, $U_{\rm ref} = U_{\rm ctr}$. δ is the plume half width based on the mean axial velocity profile $\delta = y_{0.5u}$. Experimental data are from Kotsovinos and List[1] (KL) and Ramaprian and Chandrasekhara[4] (RC).

Overall, the agreement is good, but the deviation is noticeable near the plume edge.

Using a Gaussian function for U^*, an approximate function for the mean transverse velocity can be obtained from Eq. (12) as

$$V^*(\eta) \approx \eta e^{-a\eta^2} - \frac{\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}\left(\sqrt{a}\eta\right). \tag{15}$$

The mean transverse velocity far away from the plume centerline is

$$V^*_\infty = \frac{V_\infty}{V_{\mathrm{ref}}} = -\int_0^\infty U^* d\eta \approx -\frac{\sqrt{\pi}}{2\sqrt{a}} \approx -1.06, \tag{16}$$

or

$$\frac{V_\infty}{U_{\mathrm{ref}}} = -1.06\frac{d\delta}{dx}. \tag{17}$$

Thus, the scaled boundary condition for the mean transverse flow V^*_∞ is also of $O(1)$, satisfying the requirement of the scaling patch approach.[31]

Due to its small magnitude, experimental data for V are very scarce, and the uncertainty in the measurements is significant. The approximation given by Eq. (15) is compared with experimental data of Ramaprian and Chandrasekhara[4] in Fig. 3. While Eq. (15) captures the trend of V^* well, there are noticeable differences between the experimental data and the approximation equation. More data are needed to evaluate the validity of the analysis and the approximate function.

A prominent feature of the mean transverse flow is that V^* is essentially zero around the core of the plume, and its magnitude increases monotonically toward the plume edge. The shape of the mean transverse flow in a turbulent planar plume is distinctively different from that in a turbulent plane jet, although the mean axial velocity profiles in these two free-shear turbulent flows are nearly identical.[40]

In studies of turbulent plumes, a quantity of interest is the volumetric flow rate, which can approximated from Eqs. (13) and (16) as

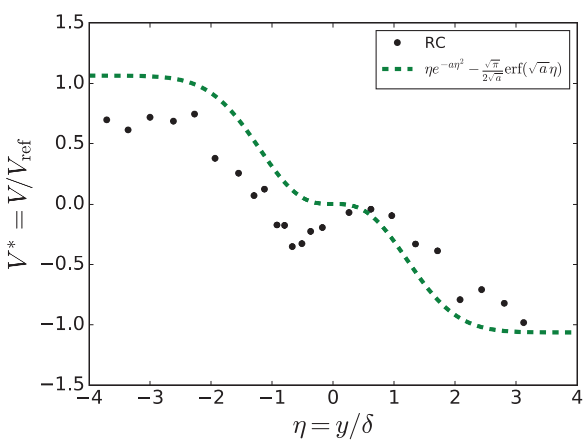

FIG. 3. Comparison of $V^* = V/V_{\mathrm{ref}}$ data with the approximation equation (15), where $V_{\mathrm{ref}} = U_{\mathrm{ctr}}\, d\delta/dx$. The experimental data are from Chandrasekhara[4] (RC).

$$\int_{-\infty}^{\infty} U^* d\eta = 2|V^*_\infty| \approx 2.12, \tag{18}$$

or in a dimensional form as

$$\int_{-\infty}^{\infty} U dy = 2|V^*_\infty| U_{\mathrm{ctr}}\delta \approx 2.12 U_{\mathrm{ctr}}\delta. \tag{19}$$

Note that in a turbulent planar plume $U_{\mathrm{ctr}} = \mathrm{const}$ and $\delta \sim x$.[15] Hence, the volumetric flow rate of the plume increases in the axial direction, arising from the entrainment of ambient fluids into the plume.

B. Admissible scaling of the mean energy equation

To transform the mean energy equation into a dimensionless form, Eq. (6c) is multiplied by $\delta/(V_{\mathrm{ref}}\Theta_{\mathrm{ref}})$ to yield

$$0 = -\left[\frac{U_{\mathrm{ctr}}}{V_{\mathrm{ref}}}\frac{\delta}{\Theta_{\mathrm{ctr}}}\frac{d\Theta_{\mathrm{ctr}}}{dx}\right]U^*\Theta^* + \eta U^*\frac{d\Theta^*}{d\eta} - V^*\frac{d\Theta^*}{d\eta} + \left[\frac{R_{v\theta,\mathrm{ref}}}{V_{\mathrm{ref}}\Theta_{\mathrm{ctr}}}\right]\frac{dR^*_{v\theta}}{d\eta}. \tag{20}$$

Physically, turbulent transport is important to balance the mean energy equation in turbulent planar plumes. Thus, an admissible scaling of the mean energy equation requires that the turbulence term in Eq. (20) must have a nominal order of magnitude 1, leading to a scale for the turbulent temperature flux as $R_{v\theta,\mathrm{ref}} = V_{\mathrm{ref}}\Theta_{\mathrm{ctr}}$.

Using results from the analysis of the mean continuity equation, it can be shown that

$$\eta U^*\frac{d\Theta^*}{d\eta} - V^*\frac{d\Theta^*}{d\eta} = \eta U^*\frac{d\Theta^*}{d\eta} - \left\{\eta U^* - \int_0^\eta U^* d\eta\right\}\frac{d\Theta^*}{d\eta} = \frac{d}{d\eta}\left(\Theta^*\int_0^\eta U^* d\eta\right) - U^*\Theta^*. \tag{21}$$

The dimensionless mean energy Eq. (20) can then be presented as

$$0 = -\left[\frac{U_{\mathrm{ctr}}}{V_{\mathrm{ref}}}\frac{\delta}{\Theta_{\mathrm{ctr}}}\frac{d\Theta_{\mathrm{ctr}}}{dx} + 1\right]U^*\Theta^* + \frac{d}{d\eta}(\Theta^*\int_0^\eta U^* d\eta) + \frac{dR^*_{v\theta}}{d\eta}. \tag{22}$$

Integrating Eq. (22) in the transverse direction from $\eta = 0$ to $\eta = \infty$ and applying boundary conditions yields an integral constraint

$$\left[\frac{U_{\mathrm{ctr}}}{V_{\mathrm{ref}}}\frac{\delta}{\Theta_{\mathrm{ctr}}}\frac{d\Theta_{\mathrm{ctr}}}{dx} + 1\right]\int_0^\infty (U^*\Theta^*)d\eta = 0. \tag{23}$$

Since $\int_0^\infty (U^*\Theta^*)d\eta$ is not zero, the integral constraint for the mean energy equation dictates that the pre-factor in Eq. (23) must be zero, or

$$\frac{U_{\mathrm{ctr}}}{V_{\mathrm{ref}}}\frac{\delta}{\Theta_{\mathrm{ctr}}}\frac{d\Theta_{\mathrm{ctr}}}{dx} = -1. \tag{24}$$

The relation between the mean temperature excess decay rate $d\Theta_{\mathrm{ctr}}/dx$ and the plume width growth $d\delta/dx$ can be obtained from Eq. (24) by substituting the definition of V_{ref}:

$$\frac{d\Theta_{\mathrm{ctr}}}{dx} = -\frac{d\delta}{dx}\frac{\Theta_{\mathrm{ctr}}}{\delta}. \tag{25}$$

As the plume width δ is proportional to x, Eq. (25) indicates that the mean temperature excess decreases inversely with the distance from the origin of the plume: $\Theta_{ctr} = (T_{ctr} - T_\infty) \sim 1/x$.[42]

Applying the integral constraint Eq. (24), the dimensionless mean energy Eq. (20) can be simplified as

$$0 = \underbrace{\left\{ U^*\Theta^* + \eta U^* \frac{d\Theta^*}{d\eta} \right\} - V^* \frac{d\Theta^*}{d\eta}}_{\mathrm{H_{adve}}} + \underbrace{\frac{dR^*_{v\theta}}{d\eta}}_{\mathrm{H_{turb}}}. \tag{26}$$

The dimensionless mean energy Eq. (26) represents the balance of two physical processes: an advective temperature transport and a turbulent temperature transport. The distributions of these two terms are illustrated in Fig. 4. Not surprisingly, the advective term has to be in balance with the turbulence term across the plume. Near the core of the plume, the advective term is positive and peaks at the plume centerline, and the turbulent term is negative. The advective term consists of one component from the axial direction and the other from the transverse direction, as shown in Fig. 4. Near the plume core, the advective heat transport is dominated by the axial component $-U^*\partial\Theta^*/x^*$, while $V^* d\Theta^*/d\eta$ is essentially zero because $V^* \approx 0$. Away from the plume core, advection in the transverse direction becomes more important.

Substituting Eq. (21) into Eq. (26), the dimensionless mean energy equation can be simplified as

$$0 = \frac{d}{d\eta}(\Theta^* \int_0^\eta U^* d\eta) + \frac{dR^*_{v\theta}}{d\eta}. \tag{27}$$

This dimensionless equation is an admissible scaling, because both terms have a nominal order of magnitude 1. Integrating Eq. (27) in the transverse direction from 0 to η and applying boundary conditions yields the solution for the turbulent heat flux $R^*_{v\theta}$ as

$$R^*_{v\theta} = -\Theta^* \int_0^\eta U^* d\eta. \tag{28}$$

It has been observed[4] that the mean temperature excess can also be approximated by a Gaussian function as

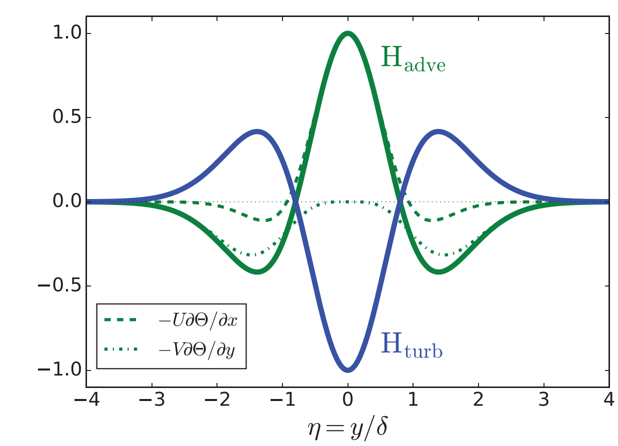

FIG. 4. Distribution of advective and turbulent terms in the mean energy Eq. (26). The curves are computed from Eq. (14) for U^*, Eq. (15) for V^*, Eq. (29) for Θ^*, and Eq. (31) for $R^*_{v\theta}$.

$$\Theta^*(\eta) \approx e^{-b\eta^2}, \tag{29}$$

where b is

$$b = \frac{\ln(2)}{(y_{0.5t}/y_{0.5u})^2}. \tag{30}$$

$y_{0.5t}$ or δ_t is the plume half-width based on the profile of the mean temperature excess, and $y_{0.5u}$ or δ is the plume half-width based on the profile of the mean axial velocity. It is observed that $y_{0.5t}$ is slightly larger than $y_{0.5u}$. Consequently, b is slightly smaller than $a = \ln(2)$. The approximate Eq. (29) for the mean excess temperature is compared with experimental data in Fig. 5, and the agreement is reasonably good.

Using Gaussian functions to approximate U^* and Θ^*, an approximate function for $R^*_{v\theta}$ can be obtained as

$$R^*_{v\theta}(\eta) \approx -\frac{\sqrt{\pi}}{2\sqrt{a}} \mathrm{erf}(\sqrt{a}\eta) e^{-b\eta^2}. \tag{31}$$

The approximate Eq. (31) for $R^*_{v\theta}$ is compared with the experimental data of Ramaprian and Chandrasekhara (RC)[4] and the simulation data of Dewan *et al.* (DKD)[9] in Fig. 6. The general trend of Eq. (31) agrees with the experimental data, but the magnitude is larger. The cause of this discrepancy is not clear to the authors. However, it is known that the measurements of temperature and $R_{v\theta}$ typically have lower accuracy and a higher level of uncertainty.[4] More data, especially high-quality simulation data, are required to check the validity of approximate Eq. (31). In the Reynolds-averaged Navier–Stokes (RANS) simulations of Dewan *et al.*,[9] three versions of the buoyancy extended $k - \epsilon - \overline{t'^2}$ modeling were used. Figure 6 shows noticeable variations among different models as well as the experimental data, and the turbulent temperature flux profile from their model M2 agrees well with Eq. (31), except near the plume edge.

Figure 6 shows that the turbulent temperature flux profile in turbulent planar plumes is anti-symmetric about the plume centerline, and possesses a prominent peak and trough. From the approximate

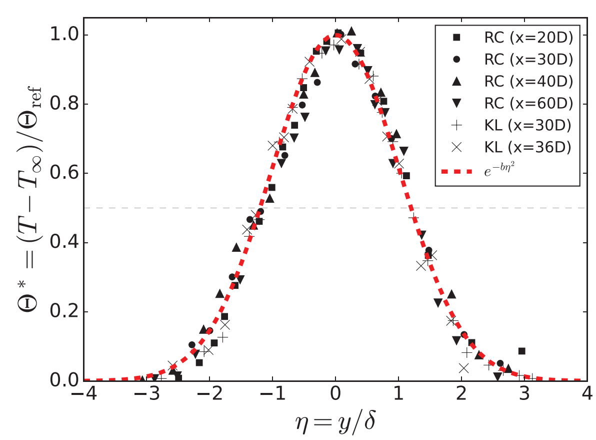

FIG. 5. Comparison of the scaled mean temperature excess data and the Gaussian Eq. (29) (dash curve). The reference temperature excess scale is $\Theta_{ref} = T_{ctr} - T_\infty$. Experimental data are from Kotsovinos and List[1] (KL) and Ramaprian and Chandrasekhara[4] (RC).

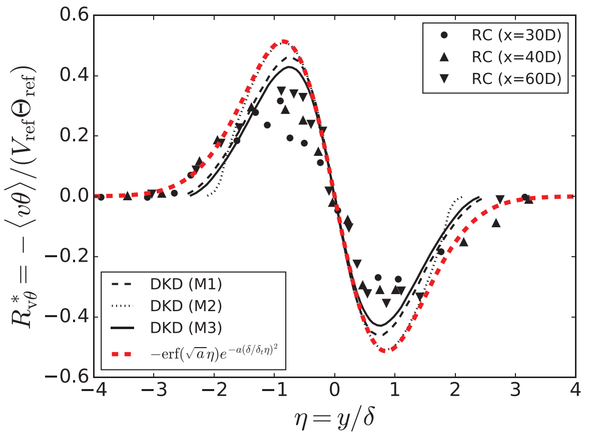

FIG. 6. Comparing the approximate equation (31) for $R^*_{v\theta}$ with experimental and numerical data. Experimental data are from Ramaprian and Chandrasekhara[4] (RC), RANS simulation with three models by Dewan *et al.*[9] (DKD).

Eq. (31), the location and magnitude of the peak and trough in the profile of the turbulent temperature flux can be estimated as

$$\eta \approx \pm 0.856, \tag{32a}$$

$$|R^*_{v\theta}|_{max} \approx 0.513. \tag{32b}$$

A quantity of interest in turbulent plumes is the heat flow rate or temperature flow rate (per unit ρc_p), which can be obtained in dimensionless form as

$$\int_{-\infty}^{\infty} (U^*\Theta^*)d\eta \approx \int_{-\infty}^{\infty} (e^{-a\eta^2} e^{-b\eta^2})d\eta \approx 1.57, \tag{33}$$

and in dimensional form as

$$\int_{-\infty}^{\infty} (U\Theta)dy \approx 1.57 U_{ctr}\Theta_{ctr}\delta. \tag{34}$$

Note that $\Theta_{ctr} \sim 1/x$ and $\delta \sim x$. Hence, the temperature flow rate in a turbulent planar plume does not vary in the axial direction; that is, it is conserved.

C. Admissible scaling of the mean axial-momentum equation

To transform the mean momentum equation into a dimensionless form, Eq. (6b) is multiplied by $\delta/(U_{ctr}V_{ref})$ to yield

$$0 = U^*\frac{dV^*}{d\eta} - V^*\frac{dU^*}{d\eta} + \left[\frac{R_{vu,ref}}{U_{ctr}V_{ref}}\right]\frac{dR^*_{vu}}{d\eta} + \left[\frac{g\beta\Theta_{ctr}\delta}{U_{ctr}V_{ref}}\right]\Theta^*. \tag{35}$$

For turbulent plumes, the turbulent force should remain important in the balance of the mean momentum equation. Thus, an admissible scaling for the mean momentum equation requires that the turbulence term in Eq. (35) must have a nominal order of magnitude 1. A proper scale for the Reynolds shear stress is then $R_{vu,ref} = U_{ctr}V_{ref}$. Accordingly, an admissible scaling of the mean momentum equation can be presented as

$$0 = U^*\frac{dV^*}{d\eta} - V^*\frac{dU^*}{d\eta} + \frac{dR^*_{vu}}{d\eta} + \left[\frac{g\beta\Theta_{ctr}\delta}{U_{ctr}V_{ref}}\right]\Theta^*. \tag{36}$$

The mean momentum Eq. (36) represents the balance of three forces: an advective force, a buoyancy force, and a turbulent force, and the distributions of the three forces are illustrated in Fig. 7. The buoyancy force is always a driving force of the plume, and the advective force is always negative, a drag force. Around the core of the plume, the turbulent force is a drag force, but near the plume edge, turbulent force acts as a driving force. The advective force consists of two components $U^*dV^*/d\eta$ and $-V^*dU^*/d\delta$, with comparable contribution, as shown in Fig. 7.

Using $dV^*/d\eta$ from Eq. (9) and V^* from Eq. (12), the advective force in Eq. (36) can be presented as

$$\begin{aligned} U^*\frac{dV^*}{d\eta} - V^*\frac{dU^*}{d\eta} &= 2U^*\frac{dV^*}{d\eta} - \frac{d(U^*V^*)}{d\eta} \\ &= -(U^*)^2 + \frac{d}{d\eta}\left(U^*\int_0^{\eta} U^* d\eta\right). \end{aligned} \tag{37}$$

Substituting Eq. (37) into Eq. (36), the dimensionless mean momentum equation can be written as

$$0 = -(U^*)^2 + \frac{d}{d\eta}\left(U^*\int_0^{\eta} U^* d\eta\right) + \frac{dR^*_{vu}}{d\eta} + \frac{g\beta\Theta_{ctr}\delta}{U_{ctr}V_{ref}}\Theta^*. \tag{38}$$

Integrating Eq. (38) in the transverse direction from $\eta = 0$ to $\eta = \infty$ and applying boundary conditions yields an integral constraint:

$$\frac{g\beta\Theta_{ctr}\delta}{U_{ctr}V_{ref}} = \frac{\int_0^{\infty}(U^*)^2 d\eta}{\int_0^{\infty}\Theta^* d\eta}. \tag{39}$$

The left side of Eq. (39) can be interpreted as a turbulence Richardson number,[39,44] defined using five basic parameters in a turbulent planar plume: $g\beta$, U_{ctr}, V_{ref}, Θ_{ctr}, and δ.

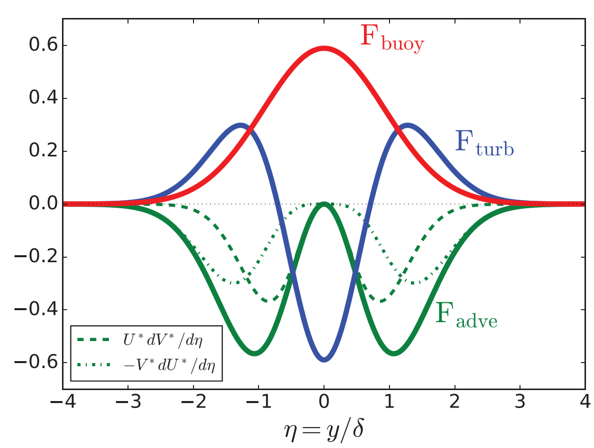

FIG. 7. Distributions of forces in the mean momentum balance Eq. (36). The curves are computed from Eq. (14) for U^*, Eq. (15) for V^*, Eq. (29) for Θ^*, and Eq. (45) for R^*_{vu}.

In the far field of turbulent planar plumes, both $\int_0^\infty (U^*)^2 d\eta$ and $\int_0^\infty \Theta^* d\eta$ are constants. Hence, the Richardson number remains invariant in the self-similar region of turbulent planar plumes. If U^* and Θ^* are approximated by Gaussian functions, the integrals in Eq. (39) become

$$\int_0^\infty (U^*)^2 d\eta \approx \frac{\sqrt{\pi}\,\mathrm{erf}\left(\sqrt{2}\sqrt{a}\eta\right)}{2^{3/2}\sqrt{a}}\Bigg|_{\eta=0}^{\eta=\infty} = \frac{\sqrt{\pi}}{2^{3/2}\sqrt{a}}, \tag{40a}$$

$$\int_0^\infty \Theta^* d\eta \approx \frac{\sqrt{\pi}\,\mathrm{erf}\left(\sqrt{b}\eta\right)}{2\sqrt{b}}\Bigg|_{\eta=0}^{\eta=\infty} = \frac{\sqrt{\pi}}{2\sqrt{b}}. \tag{40b}$$

Therefore, the Richardson number for turbulent planar plume can be approximated as

$$\frac{g\beta\Theta_{\mathrm{ctr}}\delta}{U_{\mathrm{ctr}}V_{\mathrm{ref}}} \approx \frac{1}{\sqrt{2}}\frac{\sqrt{b}}{\sqrt{a}} = \frac{1}{\sqrt{2}}\frac{y_{0.5u}}{y_{0.5t}}, \tag{41}$$

or

$$\frac{g\beta\Theta_{\mathrm{ctr}}\delta_t}{U_{\mathrm{ctr}}V_{\mathrm{ref}}} \approx \frac{1}{\sqrt{2}}. \tag{42}$$

In previous studies of plumes, the mean transverse velocity was avoided in the analysis through the integration of the mean continuity equation. Hence, V_{ref} was not present in the previous results of plumes.

Equation (42) also provides an indirect determination of the mean transverse velocity V_∞. In experimental studies of turbulent planar plumes, it is extremely challenging to obtain accurate measurements of the mean transverse velocity due to its small magnitude. On the other hand, the direct measurements of Θ_{ctr}, U_{ctr}, and δ_t can be obtained with higher level of accuracy. From Eq. (42), V_∞ can be obtained as

$$|V_\infty| \approx V_{\mathrm{ref}} \approx \frac{\sqrt{2}\,g\beta\Theta_{\mathrm{ctr}}\delta_t}{U_{\mathrm{ctr}}}. \tag{43}$$

Integrating the dimensionless mean momentum Eq. (38) yields an expression for the scaled kinematic Reynolds shear stress as

$$R^*_{\mathrm{vu}}(\eta) = \int_0^\eta (U^*)^2 d\eta - U^*\int_0^\eta U^* d\eta - \frac{g\beta\Theta_{\mathrm{ctr}}\delta}{U_{\mathrm{ctr}}V_{\mathrm{ref}}}\int_0^\eta \Theta^* d\eta. \tag{44}$$

Using Gaussian functions to approximate U^* and Θ^*, an approximate function for the kinematic Reynolds shear stress can be obtained from Eq. (44) as

$$R^*_{\mathrm{vu}}(\eta) \approx \frac{\sqrt{\pi}}{2^{3/2}\sqrt{a}}\left(\mathrm{erf}\left(\sqrt{2a}\eta\right) - \mathrm{erf}\left(\sqrt{b}\eta\right)\right) - \frac{\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}\left(\sqrt{a}\eta\right)e^{-a\eta^2}. \tag{45}$$

Empirically, Eq. (45) can be approximated by a simpler function as

$$R^*_{\mathrm{vu}}(\eta) \approx -0.606\,\mathrm{erf}\left(\sqrt{a}\eta\right)e^{-a\eta^2}. \tag{46}$$

The experimental and numerical simulation data of Reynolds shear stress are compared with Eqs. (45) and (46) in Fig. 8. The approximation equations agree reasonably well with experimental data. The RANS simulation data of Dewan *et al.*[9] show the same shape as the approximate equations, but the magnitude is slightly larger.

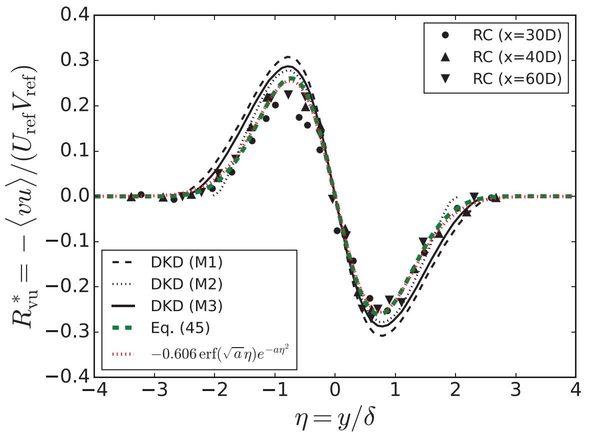

FIG. 8. Comparing the approximate equation (45) for R^*_{vu} with the experimental data of Ramaprian and Chandrasekhara[4] (RC), and RANS simulation with three models by Dewan *et al.*[9] (DKD).

Figure 8 shows that the profile of the Reynolds shear stress is also anti-symmetric about the plume centerline, and features a prominent peak and trough. The location and magnitude of the peak and trough in the profile of R^*_{vu} can be approximated from Eq. (46) as follows:

$$\eta \approx \pm 0.732, \tag{47a}$$

$$|R^*_{\mathrm{vu}}|_{\max} \approx 0.261. \tag{47b}$$

The new scaling and approximate function for the Reynolds shear stress will be useful in subsequent studies of plumes. The approximate function can also help the understanding of the flow structures in turbulent plumes.

IV. SUMMARY

In the far field of turbulent planar plumes, it is observed that the mean axial flow and mean temperature approach a self-similar state, that is, properly scaled mean axial flow or mean temperature excess profiles at different axial locations merge onto a single curve. However, previous studies of turbulent plumes failed to demonstrate a self-similar state for the normalized mean transverse flow or Reynolds shear stress profiles. This contradiction was also present in studies of other free-shear turbulent flows, such as jets, wakes, or mixing layers. One hypothesis was that there is not one universal self-similar state, and the variation of the normalized mean transverse flow in the axial direction was attributed to the memory of the initial conditions. In this paper, we show that the mean transverse flow, Reynolds shear stress, and turbulent heat flux also approach a self-similar state if properly scaled. The proper scaling is useful in presenting experimental and numerical simulation data, as well as evaluating different turbulent models for wall-free turbulence. More importantly, proper scaling is crucial to advance our understanding of the underlying physics in wall-free turbulence.

The proper scaling in turbulent planar plumes is investigated here by a relatively new scaling patch approach, originally developed for wall-bounded turbulence. Specifically, proper scales are determined by seeking admissible scaling for the mean governing equations and boundary conditions. By setting the scaled boundary conditions at the plume center to be 1, the commonly used scales for the mean axial flow and mean temperature excess are reproduced as $U_{\mathrm{ref}} = U_{\mathrm{ctr}}$ and $\Theta_{\mathrm{ref}} = \Theta_{\mathrm{ctr}}$, respectively. From an admissible scaling of the mean continuity equation, a proper scale for the mean transverse flow is found as $V_{\mathrm{ref}} = (d\delta/dx)U_{\mathrm{ctr}}$. Similarly, a proper scale for the Reynolds shear stress is found as $R_{\mathrm{vu,ref}} = U_{\mathrm{ctr}} V_{\mathrm{ref}}$ from an admissible scaling of the mean momentum equation, and a proper scaling for the turbulent temperature flux is found as $R_{\mathrm{v}\theta,\mathrm{ref}} = V_{\mathrm{ref}}\Theta_{\mathrm{ctr}}$ from an admissible scaling of the mean energy equation. Analytical and approximate equations are developed for the scaled mean transverse flow, Reynolds shear stress, and turbulent heat flux and are found to agree favorably with experimental and numerical data. These approximate equations are useful in subsequent experimental and numerical studies of turbulent planar plumes. For the reader's convenience, the analysis results of turbulent planar plumes are summarized in Table III.

The present analysis reveals the critical role of the mean transverse flow in the scaling of the flow and heat transport in a turbulent planar plume. In previous studies of turbulent free-shear flows, the scale for the mean transverse flow rarely received any attention. The mean continuity equation is typically integrated to remove the mean transverse flow from the analysis. However, recent studies of wall-bounded turbulent flow[45,46] and turbulent planar jet[40] have revealed that the mean transverse velocity scale is critical in the proper scaling of the mean momentum equations. In this work, we show that the proper scale for the Reynolds shear stress is a mix of the scales for the mean axial and transverse flows, $U_{\mathrm{ctr}} V_{\mathrm{ref}}$. A proper scale for the turbulent temperature flux is also a mix of scales for the mean transverse flow and mean temperature excess, $V_{\mathrm{ref}}\Theta_{\mathrm{ctr}}$. Thus, the mean transverse velocity scale is essential in the scaling of the mean flow and heat transport in turbulent plumes.

The present work reveals striking similarities between turbulent planar plumes and other free-shear turbulence, such as jets[40] and the outer layer of wall-bounded turbulence.[45,47] However, there are also important differences between turbulent plumes and other free-shear turbulence, for example, the existence of buoyancy force in the mean momentum equation for turbulent plumes. The influence of buoyancy force in turbulent plumes can be quantified by a Richardson number $\mathrm{Ri} = (g\beta\Theta_{\mathrm{ctr}}\delta_t)/(U_{\mathrm{ctr}} V_{\mathrm{ref}})$, defined by the key parameters of turbulent plumes: the buoyancy parameter $g\beta$, the bulk temperature difference Θ_{ctr}, the plume width δ_t, and velocity scales for the mean axial and transverse flows. From the global integral of the mean momentum equation, the Richardson number is found to be a constant $\mathrm{Ri} \approx 1/\sqrt{2}$ once the turbulent plume reaches a self-similar state. This Richardson number arises naturally from the scaling patch analysis of the mean momentum equation and directly reflects the ratio between the buoyancy and turbulence forces in a turbulent plume. Thus, this Richardson number is closely related to the underlying physics in a turbulent plume. The discovery of this Richardson number demonstrates the power of the scaling patch approach, which unifies the analysis of the pressure- or shear-driven wall-bounded turbulence and buoyancy-driven wall-free turbulent plumes.

TABLE III. Summary of turbulent planar plume results.

Mean axial velocity scale	$U_{\mathrm{ref}} = U_{\mathrm{ctr}}$
Mean transverse velocity scale	$V_{\mathrm{ref}} = \dfrac{d\delta}{dx} U_{\mathrm{ctr}}$
Mean temperature excess scale	$\Theta_{\mathrm{ref}} = T_{\mathrm{ctr}} - T_{\infty}$
Reynolds shear stress scale	$R_{\mathrm{vu,ref}} = U_{\mathrm{ctr}} V_{\mathrm{ref}} = \dfrac{d\delta}{dx} U_{\mathrm{ctr}}^2$
Turbulent temperature flux scale	$R_{\mathrm{v}\theta,\mathrm{ref}} = V_{\mathrm{ref}}\Theta_{\mathrm{ctr}} = \dfrac{d\delta}{dx} U_{\mathrm{ctr}}(T_{\mathrm{ctr}} - T_{\infty})$
Mean continuity equation	$0 = -\eta \dfrac{dU^*}{d\eta} + \dfrac{dV^*}{d\eta}$
Mean momentum equation	$0 = U^* \dfrac{dV^*}{d\eta} - V^* \dfrac{dU^*}{d\eta} + \dfrac{dR_{\mathrm{vu}}^*}{d\eta} + \dfrac{g\beta\Theta_{\mathrm{ctr}}\delta}{U_{\mathrm{ctr}} V_{\mathrm{ref}}}\Theta^*$
Mean energy equation	$0 = U^*\Theta^* + \eta U^* \dfrac{d\Theta^*}{d\eta} - V^* \dfrac{d\Theta^*}{d\eta} + \dfrac{dR_{\mathrm{v}\theta}^*}{d\eta}$
Equation for V^*	$V^* = \eta U^* - \int_0^\eta U^* d\eta \approx \eta e^{-a\eta^2} - \dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}\mathrm{erf}(\sqrt{a}\eta)$
Equation for R_{vu}^*	$R_{\mathrm{vu}}^* = \int_0^\eta (U^*)^2 d\eta - U^* \int_0^\eta U^* d\eta - \dfrac{g\beta\Theta_{\mathrm{ctr}}\delta}{U_{\mathrm{ctr}} V_{\mathrm{ref}}} \int_0^\eta \Theta^* d\eta$
Equation for $R_{\mathrm{v}\theta}^*$	$R_{\mathrm{v}\theta}^* = -\Theta^* \int_0^\eta U^* d\eta \approx -\dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}\mathrm{erf}(\sqrt{a}\eta) e^{-b\eta^2}$
Volumetric flow rate	$\int_{-\infty}^{\infty} U^* d\eta = 2\vert V_\infty^*\vert$ or $\int_{-\infty}^{\infty} U dy \approx 2.12 U_{\mathrm{ctr}}\delta.$
Temperature flow rate	$\int_{-\infty}^{\infty} (U^*\Theta^*) d\eta \approx 1.57$ or $\int_{-\infty}^{\infty} (U\Theta) dy \approx 1.57 U_{\mathrm{ctr}}\Theta_{\mathrm{ctr}}\delta$

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created in this study.

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