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ABSTRACT

Proper scaling for the mean transverse flow and Reynolds shear stress in a turbulent plane jet is determined using a scaling patch approach. By seeking an admissible scaling, a key concept in the scaling patch approach, for the mean continuity equation, a proper scale for the mean transverse flow in a turbulent plane jet is found as $V_{\text{ref}} = -\delta dU_{\text{ctr}}/dx$, where δ is the jet half width and dU_{ctr}/dx is the decay rate of the mean axial velocity at the jet centerline. By seeking an admissible scaling for the mean axial momentum equation, a proper scale for the kinematic Reynolds shear stress is found as $R_{uv,\text{ref}} = U_{\text{ctr}} V_{\text{ref}}$, which is a mix of the velocity scales in the axial and transverse directions. Approximation functions for the scaled mean transverse flow and Reynolds shear stress are developed and found to agree well with experimental and numerical data. Similarities and differences between the scales of the mean transverse flow and Reynolds shear stress in turbulent plane jets and zero-pressure-gradient turbulent boundary layer flows are clarified.

I. INTRODUCTION

Free shear flow occurs when there are no solid walls interacting directly with the flow,[1] as in jets, plumes, wakes, and mixing layers; it is encountered in a broad range of geophysical, environmental, and engineering flows. Free shear flows are slender, i.e., they spread slowly in the transverse direction compared to the advection in the streamwise direction. Hence, scatter is significant in experimental measurements of the transverse flow, and confusion exists to this day about the proper scaling of the mean transverse flow. In this paper, we apply a relatively new scaling patch approach to determine the proper scales for the mean transverse flow and the Reynolds shear stress in turbulent plane jets.

As sketched in Fig. 1, a turbulent plane jet is a canonical free shear turbulent flow, and this flow has been studied for nearly a century, analytically and experimentally, and recently by numerical simulation.[1–24] Exiting the nozzle, the jet decays in the axial direction and at the same time spreads transversely. A common measure of the jet width is the so-called half-value width $y_{1/2}$, i.e., the local distance of the point with half the mean centerline velocity $U|_{y=y_{1/2}} = 0.5U_{\text{ctr}}$. At different locations in the axial direction, the transverse distributions of mean flow are different. However, in the far field of a jet flow, it has been observed that the profiles of mean axial flow become self-similar; the mean axial velocity profiles at different axial locations merge well onto a single curve if plotted as U/U_{ctr} vs $\eta = y/y_{1/2}$,[1,2,7,25] as shown in Fig. 2. Similar plots can be found in the standard texts on turbulent flow, including Ref. 1. The scaled mean axial velocity can be approximated by a Gaussian function $U/U_{\text{ctr}} \approx e^{-a\eta^2}$, as represented by the dashed curve, with the coefficient a having the standard value, i.e., $a = \ln(2) \approx 0.693$. The self-similarity of the mean axial velocity is convenient in the presentation of experimental or numerical turbulent plane jet data. More importantly, self-similarity is a concept of fundamental significance in the study of turbulent flows.[10,17,21,26,27] However, there remain debates on the universality of self-similarity in free shear turbulence, including turbulent plane jets.

Traditional analysis of free-shear turbulence, including turbulent jets, can be found in the book by Tennekes and Lumley,[25] and it typically starts with a self-similarity assumption, followed by definitions of the self-similar mean axial flow with a function f and self-similar Reynolds shear stress with a function g. The mean continuity equation was integrated to eliminate the mean transverse velocity from the analysis, and the traditional analysis resulted in an ordinary differential equation for the mean momentum equation. In early studies, an eddy

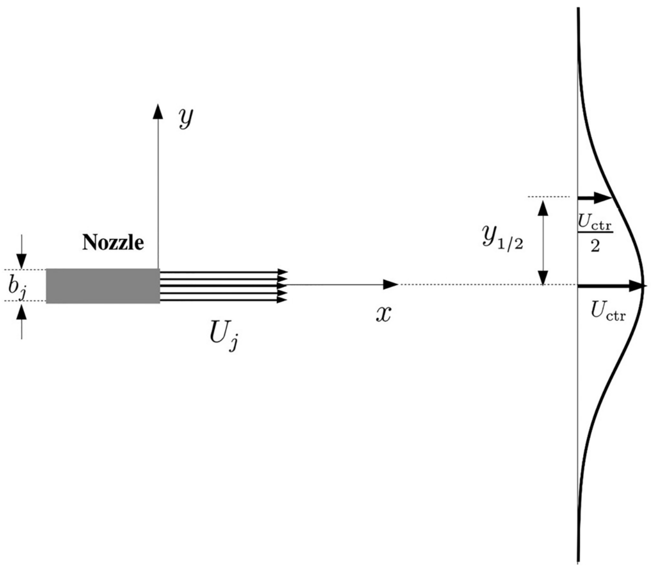

FIG. 1. Geometry and coordinate system of a plane jet. The axial direction is denoted as x and the transverse direction as y. The height of the jet nozzle is b_j, and the jet exit velocity is denoted as U_j. The mean velocity at the centerline is denoted as U_{ctr}.

viscosity model was often used to close the Reynolds shear stress in order to obtain approximate solutions. In the traditional analysis presented by Tennekes and Lumley, one velocity scale U_{ctr} was used to describe the flow, and the Reynolds shear stress was normalized by U_{ctr}^2. In the analyses of George[10] and Cafiero and Vassilicos,[21] the Reynolds shear stress was normalized by a separate scale, not U_{ctr}^2. In traditional analyses of turbulent jets, the scaling of the mean transverse flow was not explicitly determined, and the relation among the scaling

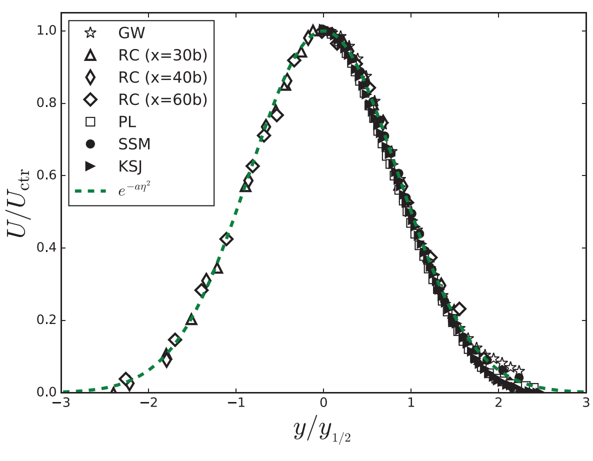

FIG. 2. Mean axial velocity and approximate expression using a Gaussian function (dashed curve), $e^{-a\eta^2}$, where $a = \ln(2)$. Experimental data (open symbols) are from Gutmark and Wygnanski (GW),[28] Ramaprian and Chandrasekhara (RC),[8] and Panchapakesan and Lumley (PL).[11] Direct numerical simulation data (solid symbols) are from Stanley, Sarkar, and Mellado (SSM),[12] and Klein, Sadiki, and Janicka (KSJ).[13] To avoid clutter, DNS data are only plotted for one half of the jet.

of the mean axial flow, mean transverse flow, and Reynolds shear stress was not clear.

The goal of this paper is to seek proper scaling of the mean transverse flow and Reynolds shear stress using a relatively new scaling patch approach,[29–31] which was developed originally for shear-driven wall-bounded turbulence. A key concept in the scaling patch approach is the admissible scaling, which requires that the scaled governing equations have at least two terms with a nominal order of magnitude 1, and the scaled boundary conditions should also be zero or a nominal order of magnitude 1. The governing equations can be scaled in any number of ways, creating an infinite number of versions of dimensionless equations, and all versions are mathematically equivalent, i.e., one scaling can be transformed to another by simple rescaling factors.[30,31] However, in different regions of the flow, the relative magnitude of the terms may be different, and the balance of terms in each transport equation might be different. The admissible scaling helps reveal the relative magnitude of terms, which corresponds to each distinct dynamical regime for each transport equation. Here, we show that the admissible scaling can clearly reveal the relative magnitude of terms in the mean governing equations of turbulent plane jets, which in turn assists in determining the proper scaling of the mean transverse flow and Reynolds shear stress. This clarifies some missing parts in traditional analyses. The properly scaled mean transverse flow and Reynolds shear stress profiles from different axial locations are shown to merge well onto a single curve; that is to say, they approach a self-similar state in the far field.

The rest of the paper is organized as follows. In Sec. II, the mean continuity and mean axial momentum equation are presented. Proper scales for the mean axial flow, mean transverse flow, and Reynolds shear stress are determined by seeking admissible scaling for the mean continuity equation, the mean momentum equation, and the boundary conditions. Approximate equations are developed for the mean transverse flow and the Reynolds shear stress and found to agree well with experimental and numerical data. Section III discusses the similarity and difference in the scaling between the turbulent plane jet and the zero-pressure-gradient turbulent boundary layer. Section IV summarizes the work.

II. ANALYSIS OF THE GOVERNING EQUATIONS

As a jet evolves much more slowly in the axial direction than the transverse direction, Prandtl's boundary layer approximation equations are commonly used for turbulent plane jets.[1,2] The mean continuity equation and the mean momentum equation in the axial direction are

$$0 = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}, \tag{1a}$$

$$0 = -U\frac{\partial U}{\partial x} - V\frac{\partial U}{\partial y} + \nu\frac{\partial^2 U}{\partial y^2} + \frac{\partial R_{uv}}{\partial y}. \tag{1b}$$

In this paper, an upper case letter denotes a mean velocity component and a lower case letter denotes its fluctuation. For example, U is the mean axial velocity and u is the velocity fluctuation in the axial direction. V is the mean transverse velocity, and v is the velocity fluctuation in the transverse direction. The kinematic Reynolds shear stress is denoted as $R_{uv} = -\langle uv \rangle$, where angle brackets denote Reynolds averaging. The boundary conditions for the turbulent plane jet are listed in

Table I. Note that the mean transverse velocity and Reynolds shear stress are zero at the jet centerline due to symmetry.

To transform the governing equation into a self-similar form, the variables are normalized by reference scales and denoted as follows:

$$\eta \stackrel{\text{def}}{=} \frac{y}{\delta(x)}; \quad U^*(\eta) \stackrel{\text{def}}{=} \frac{U(x,y)}{U_{\text{ref}}(x)}; \quad V^*(\eta) \stackrel{\text{def}}{=} \frac{V(x,y)}{V_{\text{ref}}(x)}; \quad R^*_{uv}(\eta) \stackrel{\text{def}}{=} \frac{R_{uv}(x,y)}{R_{uv,\text{ref}}(x)}, \tag{2}$$

where δ is a measure of the jet width and $U_{\text{ref}}, V_{\text{ref}}, R_{uv,\text{ref}}$ are proper scales for the mean axial velocity, the mean transverse velocity, and the kinematic Reynolds shear stress, respectively. Note that in many of the previous studies of turbulent plane jets, only one velocity scale, the jet centerline velocity U_{ctr}, is used for all the mean flow variables and turbulent statistics. Here, we specify different reference scales for U, V, and R_{uv}.

We first note that the derivatives of η with respect to x and y are

$$\frac{\partial \eta}{\partial x} = -\frac{1}{\delta}\frac{d\delta}{dx}\eta, \tag{3a}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{\delta}. \tag{3b}$$

As a result, the derivative of U with respect to x is

$$\begin{aligned}\frac{\partial U}{\partial x} &= \frac{dU_{\text{ref}}}{dx}U^* - \frac{U_{\text{ref}}}{\delta}\frac{d\delta}{dx}\eta\frac{dU^*}{d\eta}\\ &= \left\{\frac{dU_{\text{ref}}}{dx} + \frac{U_{\text{ref}}}{\delta}\frac{d\delta}{dx}\right\}U^* - \frac{U_{\text{ref}}}{\delta}\frac{d\delta}{dx}\frac{d(\eta\, U^*)}{d\eta}.\end{aligned} \tag{4}$$

The derivatives of U, V, and R_{uv} with respect to y are

$$\frac{\partial U}{\partial y} = \frac{U_{\text{ref}}}{\delta}\frac{dU^*}{d\eta}, \tag{5a}$$

$$\frac{\partial V}{\partial y} = \frac{V_{\text{ref}}}{\delta}\frac{dV^*}{d\eta}, \tag{5b}$$

$$\frac{\partial R_{uv}}{\partial y} = \frac{R_{uv,\text{ref}}}{\delta}\frac{dR^*_{uv}}{d\eta}. \tag{5c}$$

Substituting the self-similar variables U^*, V^* and their derivatives, the continuity equation (1a) and the mean momentum equation (1b) can be presented as

$$0 = \left\{\frac{dU_{\text{ref}}}{dx} + \frac{U_{\text{ref}}}{\delta}\frac{d\delta}{dx}\right\}U^* - \frac{U_{\text{ref}}}{\delta}\frac{d\delta}{dx}\frac{d(\eta\, U^*)}{d\eta} + \frac{V_{\text{ref}}}{\delta}\frac{dV^*}{d\eta}, \tag{6a}$$

$$\begin{aligned}0 = \frac{U_{\text{ref}}V_{\text{ref}}}{\delta}U^*\frac{dV^*}{d\eta} - \frac{U_{\text{ref}}V_{\text{ref}}}{\delta}V^*\frac{dU^*}{d\eta} + \frac{\nu U_{\text{ref}}}{\delta^2}\frac{d^2U^*}{d\eta^2}\\ + \frac{R_{uv,\text{ref}}}{\delta}\frac{dR^*_{uv}}{d\eta}.\end{aligned} \tag{6b}$$

TABLE I. Boundary conditions in the turbulent plane jet.

$y=0$	$U = U_{\text{ctr}},\ V = 0,\quad R_{uv} = 0$
$y=\infty$	$U = 0,\quad V = V_\infty,\ R_{uv} = 0$

The corresponding boundary conditions for the normalized variables are listed in Table II.

Next, the reference scales U_{ref}, V_{ref}, and $R_{\text{uv,ref}}$ will be obtained by seeking admissible scaling for the mean continuity equation, the mean momentum equation, and the normalized boundary conditions. Admissible scaling is a key concept in the scaling patch analysis, a relatively new approach originally developed by Fife and co-workers for shear-driven wall-bounded turbulence.[29–31] The scaling patch approach has been applied to passive scalar transport in a turbulent pipe or channel flow,[32,33] turbulent boundary flow with roughness,[34] turbulent Taylor–Couette flow,[35] and buoyancy-driven turbulence convection.[36,37]

It is well known that, in the far field of turbulent plane jets, the flowfield can be characterized by a single length scale, i.e., the jet width. Here, we show that a concept of admissible scaling in the scaling patch approach can also be applied to free shear turbulence, as in a turbulent plane jet, to reveal proper scaling of the mean flow. To be an admissible scaling, the scaled equations must have at least two terms with a nominal order of magnitude 1. Moreover, the properly scaled boundary conditions should be either zero or of $O(1)$. Therefore, a natural scale for the mean axial velocity is $U_{\text{ref}} = U_{\text{ctr}}$ because the scaled boundary condition at the channel centerline becomes $U^* = 1$ (see Table II).

A. Admissible scaling for the continuity equation

To obtain a dimensionless mean continuity equation, multiplying δ/V_{ref} by Eq. (6a) produces

$$0 = \left[\frac{\delta}{V_{\text{ref}}}\frac{dU_{\text{ctr}}}{dx} + \frac{U_{\text{ctr}}}{V_{\text{ref}}}\frac{d\delta}{dx}\right]U^* - \left[\frac{U_{\text{ctr}}}{V_{\text{ref}}}\frac{d\delta}{dx}\right]\frac{d(\eta\, U^*)}{d\eta} + \frac{dV^*}{d\eta}. \tag{7}$$

For brevity, we introduce a ratio A defined as (its value is determined in Sec. II B)

$$A \stackrel{\text{def}}{=} \frac{\dfrac{U_{\text{ctr}}}{V_{\text{ref}}}\dfrac{d\delta}{dx}}{\dfrac{\delta}{V_{\text{ref}}}\dfrac{dU_{\text{ctr}}}{dx}} = \frac{U_{\text{ctr}}}{\delta}\frac{\dfrac{d\delta}{dx}}{\dfrac{dU_{\text{ctr}}}{dx}}. \tag{8}$$

In studies of turbulent jets, power laws are often used to approximate the growth of the jet width and the decay of the jet centerline velocity. Using a power law for the jet width as $\delta \approx c_1x^n$ and a power law for the decay of the jet centerline velocity as $U_{\text{ctr}} \approx c_2x^m$, it can be easily shown that A will be a bounded constant. In Sec. II B [see Eq. (26)], the ratio A is shown to be $A = -2$, indeed a constant of order 1.

Using the notation A, the dimensionless continuity equation can be presented as

$$0 = \left[\frac{\delta}{V_{\text{ref}}}\frac{dU_{\text{ctr}}}{dx}\right]\left\{(1+A)U^* - A\frac{d(\eta\, U^*)}{d\eta}\right\} + \frac{dV^*}{d\eta}. \tag{9}$$

TABLE II. Boundary conditions for normalized variables in the turbulent plane jet.

$\eta=0$	$U^* = \frac{U_{\text{ctr}}}{U_{\text{ref}}},\ V^* = 0,\quad R^*_{uv} = 0$
$\eta=\infty$	$U^* = 0,\quad V^* = \frac{V_\infty}{V_{\text{ref}}},\ R^*_{uv} = 0$

For Eq. (9) to be an admissible scaling, we can set the mean transverse velocity scale as

$$V_{\mathrm{ref}} = -\delta \frac{dU_{\mathrm{ctr}}}{dx}. \tag{10}$$

The definition of V_{ref} in Eq. (10) has a negative sign because the mean axial velocity at the centerline decreases in the axial direction ($dU_{\mathrm{ctr}}/dx < 0$). Using the definition of A in Eq. (8), the mean transverse velocity scale can also be presented as

$$V_{\mathrm{ref}} = -\frac{U_{\mathrm{ctr}}}{A}\frac{d\delta}{dx}. \tag{11}$$

Substituting the definition of V_{ref} in Eq. (10), an admissible continuity equation can be written as

$$0 = -(1+A)U^* + A\frac{d(\eta U^*)}{d\eta} + \frac{dV^*}{d\eta}. \tag{12}$$

Integrating Eq. (12) in the transverse direction from 0 to η and applying boundary conditions yield the exact solution for the scaled mean transverse velocity as

$$V^* = -A\eta U^* + (1+A)\int_0^\eta U^* d\eta. \tag{13}$$

It has been observed, as shown in Fig. 2, that the normalized mean axial velocity can be approximated by a Gaussian function: $U^*(\eta) = e^{-a\eta^2}$, where $a = \ln(2)$ is due to the definition of $U^*(\eta = 1) = 0.5$. Using the Gaussian function for U^*, the integral of the mean axial velocity is

$$\int_0^\eta U^* d\eta = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,\mathrm{erf}\big(\sqrt{a}\eta\big). \tag{14}$$

An approximate function for V^* is, then, obtained as

$$V^*(\eta) \approx -A\eta e^{-a\eta^2} + (1+A)\frac{1}{2}\sqrt{\frac{\pi}{a}}\,\mathrm{erf}\big(\sqrt{a}\eta\big). \tag{15}$$

The ratio A is determined as $A = -2$ by an integral constraint of the mean momentum equation [see Eq. (26) in Sec. II B]. Thus, an approximate function for V^* is

$$V^*(\eta) \approx 2\eta e^{-a\eta^2} - \frac{1}{2}\sqrt{\frac{\pi}{a}}\,\mathrm{erf}\big(\sqrt{a}\eta\big). \tag{16}$$

Note that the magnitude of the mean transverse flow reaches a maximum value at the jet edge and $|V|_{\max}/V_{\mathrm{ref}} = |V^*_\infty| \approx 1.06$. Thus, the scaled boundary condition for V^* is also of an order 1, satisfying the requirement of admissible scaling. Therefore, the mean transverse velocity outside the turbulent plane jet V_∞ can also be used as a reference velocity scale for the mean transverse flow. The mean transverse velocity normalized by V_∞ can be approximated as

$$\frac{V}{|V_\infty|} \approx 1.89\eta e^{-a\eta^2} - \frac{1}{2.12}\sqrt{\frac{\pi}{a}}\,\mathrm{erf}\big(\sqrt{a}\eta\big). \tag{17}$$

Within the jet core, the mean transverse velocity profile possesses a local maximum/minimum, and its location and value are

$$\eta = \pm\frac{1}{2}\frac{1}{\sqrt{a}} \approx \pm 0.6, \tag{18a}$$

$$V^* \approx \pm 0.38. \tag{18b}$$

Figure 3 compares experimental and numerical data of the mean transverse velocity with the approximate equation [Eq. (16)]. Experimental data are from Gutmark and Wygnanski (GW)[28] and Ramaprian and Chandrasekhara (RC).[8] Direct numerical simulation (DNS) data are from Stanley, Sarkar, and Mellado (SSM)[12] and Klein, Sadiki, and Janicka (KSJ).[13] The Reynolds number in the GW experimental study is $Re_j \overset{\mathrm{def}}{=} U_j b_j/\nu \approx 30\,000$, where U_j is the jet exit velocity, b_j is the exit slot height, and ν is the kinematic viscosity. The Reynolds number in the numerical studies of SSM and KSJ is much lower, at $Re_j \cong 3\,000$. The DNS data agree well with the approximation equation [Eq. (16)]. The deviation between the data and the approximate function near the edge of the jet in Fig. 3 is likely caused by (1) the uncertainty in the measurement of V and (2) the deficiency of using a Gaussian function to approximate the U profile near the edge of the turbulent plane jet.

The mean transverse flow in experimental studies of turbulent plane jets is very small. For example, in the experiment of Ramaprian and Chandrasekhara,[8] the mean transverse velocity was about $0.5 - 1$ cm/s. Therefore, it was extremely challenging to obtain the accurate measurement of the mean transverse flow, and the uncertainty was rather large, as shown by the scatter in Fig. 3. Nevertheless, the experimental data are close to the approximation equation [Eq. (16)]. For example, at $\eta = -3$, the difference between the measurement and the approximation equation is about 20%, within the uncertainty of experimental measurements. In theory, the mean transverse velocity is anti-symmetric about the jet centerline. The deviation of experimental data from anti-symmetry in Fig. 3 is attributed to measurement uncertainty.

In the work by Cafiero and Vassilicos,[21] the mean transverse velocity was scaled by $U_{\mathrm{ctr}}\, d\delta/dx$, and that scale is similar to V_{ref} proposed here. In this paper, an explicit approximation function is developed for the scaled mean transverse velocity.

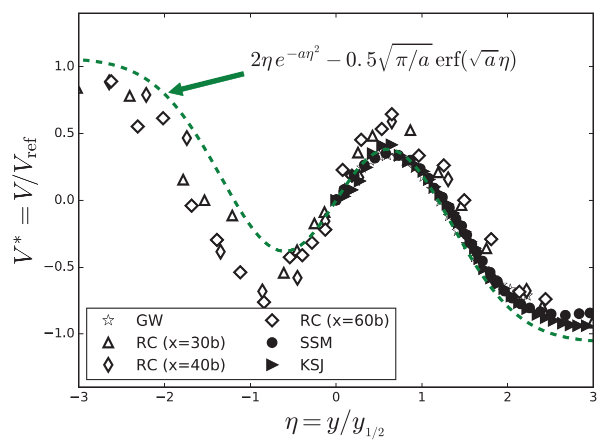

FIG. 3. Scaled mean transverse velocity $V/V_{\mathrm{ref}} = V/(-\delta dU_{\mathrm{ctr}}/dx)$ and the approximate equation [Eq. (16)]. Data Refs. 8, 12, 13, and 28 as in Fig. 2.

In studies of turbulent jets, a quantity of interest is the volumetric flow rate, which can be obtained by integrating Eq. (12) from $\eta = -\infty$ to $\eta = \infty$ as

$$\int_{-\infty}^{\infty} U^* d\eta = \frac{V^*_{\infty} - V^*_{-\infty}}{1+A} = \frac{2V^*_{\infty}}{1+A} = -2V^*_{\infty} \approx 2.12. \qquad (19)$$

Note that the mean transverse flow in a turbulent plane jet is anti-symmetric about the centerline, i.e., $V^*_{-\infty} = -V^*_{\infty}$. Thus, the volumetric flow rate in a turbulent plane jet is directly related to the mean transverse velocity at the edge of the jet. In the dimensional form, the volumetric flow rate is

$$\int_{-\infty}^{\infty} U dy \approx 2.12 U_{\rm ctr}\delta. \qquad (20)$$

For a turbulent plane jet, it is known that $\delta \sim x$ and $U_{\rm ctr} \sim 1/x^{0.5}$ [e.g., see Ref. 1]. Thus, the volumetric flow rate of the turbulent plane jet increases in the axial direction, arising from the entrainment of ambient fluid into the jet.

B. Admissible scaling for the mean axial momentum equation

To transform the mean momentum equation into a dimensionless form, multiplying $\delta/(U_{\rm ref}V_{\rm ref})$ to Eq. (6b) produces

$$0 = U^*\frac{dV^*}{d\eta} - V^*\frac{dU^*}{d\eta} + \left[\frac{\nu}{\delta V_{\rm ref}}\right]\frac{d^2U^*}{d\eta^2} + \left[\frac{R_{uv,\rm ref}}{U_{\rm ref}V_{\rm ref}}\right]\frac{dR^*_{uv}}{d\eta}. \qquad (21)$$

At a sufficiently high Reynolds number, the prefactor to the viscous force is small, $\nu/(\delta V_{\rm ref}) \ll 1$, meaning that the viscous force is negligible. Therefore, the force balance in a turbulent plane jet is between the advective force (the first two terms) and the turbulent force (the last term). To make Eq. (21) an admissible scaling, the reference Reynolds shear stress has to be set as $R_{uv,\rm ref} = U_{\rm ref}V_{\rm ref}$. Neglecting the viscous force, the admissible scaling of the mean momentum equation becomes

$$0 = U^*\frac{dV^*}{d\eta} - V^*\frac{dU^*}{d\eta} + \frac{dR^*_{uv}}{d\eta}. \qquad (22)$$

Applying $dV^*/d\eta$ obtained from Eq. (12), simple mathematical manipulation transforms the advective term in Eq. (22) as follows:

$$\begin{aligned} U^*\frac{dV^*}{d\eta} - V^*\frac{dU^*}{d\eta} &= 2U^*\frac{dV^*}{d\eta} - \frac{d(U^*V^*)}{d\eta} \\ &= 2U^*\left\{(1+A)U^* - A\frac{d(\eta U^*)}{d\eta}\right\} - \frac{d(U^*V^*)}{d\eta} \\ &= (2+A)(U^*)^2 - A\frac{d(\eta (U^*)^2)}{d\eta} - \frac{d(U^*V^*)}{d\eta}. \end{aligned} \qquad (23)$$

Thus, the dimensionless mean momentum Eq. (22) can be presented as

$$0 = (2+A)(U^*)^2 - A\frac{d(\eta (U^*)^2)}{d\eta} - \frac{d(U^*V^*)}{d\eta} + \frac{dR^*_{uv}}{d\eta}. \qquad (24)$$

Integrating Eq. (24) from $\eta = 0$ to $\eta = \infty$ and applying boundary conditions yield

$$0 = (2+A)\int_0^{\infty} (U^*)^2 d\eta. \qquad (25)$$

For Eq. (25) to be valid, the prefactor has to be zero,

$$A = -2, \qquad (26a)$$

or

$$\frac{d\delta}{dx} = -2\frac{\delta}{U_{\rm ctr}}\frac{dU_{\rm ctr}}{dx}. \qquad (26b)$$

Thus, the integral of the mean axial momentum equation provides a relation between the jet width growth rate and the jet centerline velocity decay rate. Integrating Eq. (26b) provides the integral constraint on the self-similar plane-jet flows as

$$U^2_{\rm ctr}(x)\delta(x) = \text{const}. \qquad (27)$$

This relation [Eq. (27)] is not new but was derived previously by Townsend,[7] George,[10] and Cafiero and Vassilicos.[21] Note that the integral constraint does not dictate a specific function form for $U_{\rm ctr}(x)$ or $\delta(x)$, individually, but it does require the product of $U^2_{\rm ctr}(x)\delta(x)$ to be a constant when a jet reaches a self-similar state.

Applying the constraint [Eq. (26)], the mean axial momentum [Eq. (24)] can be simplified as

$$0 = -A\frac{d(\eta (U^*)^2)}{d\eta} - \frac{d(U^*V^*)}{d\eta} + \frac{dR^*_{uv}}{d\eta}. \qquad (28)$$

Integrating Eq. (28) in the transverse direction from 0 to η, applying boundary conditions and substituting Eq. (13) for V^* yield an equation for the Reynolds shear stress as

$$R^*_{uv} = A\eta(U^*)^2 + U^*V^* = (1+A)U^*\int_0^{\eta} U^* d\eta = -U^*\int_0^{\eta} U^* d\eta. \qquad (29)$$

Using a Gaussian function to approximate U^*, the scaled Reynolds shear stress can be approximated as

$$R^*_{uv} \approx -\frac{\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}\left(\sqrt{a}\eta\right)e^{-a\eta^2}. \qquad (30)$$

The peak magnitude of the scaled Reynolds shear stress occurs at $\eta \approx \pm 0.75$, and the peak value is $|R^*_{uv}|_{\max} \approx 0.45$. Figure 4 shows that the experimental and numerical data of the Reynolds shear stress agree well with the approximation equation [Eq. (30)]. The deviation in the DNS data of Klein *et al.* is likely caused by the low Reynolds number effect.

In textbooks and previous papers on turbulent jets, Reynolds shear stress data are typically scaled by $U^2_{\rm ctr}$. George[10] proposed a different scale for the Reynolds shear stress as $U^2_{\rm ctr}\, d\delta/dx$. The scale proposed here $U_{\rm ctr}(-\delta\, dU_{\rm ctr}/dx)$ is related to the George scale by the factor A. The novelty of the present work is that the scaling is derived from the relatively new scaling patch approach. Moreover, an explicit approximation function is developed for the scaled Reynolds shear stress.

Approximating U^* by a Gaussian function, the kinematic momentum flow rate in a turbulent plane jet can be approximated as

$$\int_{-\infty}^{\infty} (U^*)^2 d\eta \approx \frac{1}{2^{1/2}}\sqrt{\frac{\pi}{a}} \approx 1.5 \qquad (31)$$

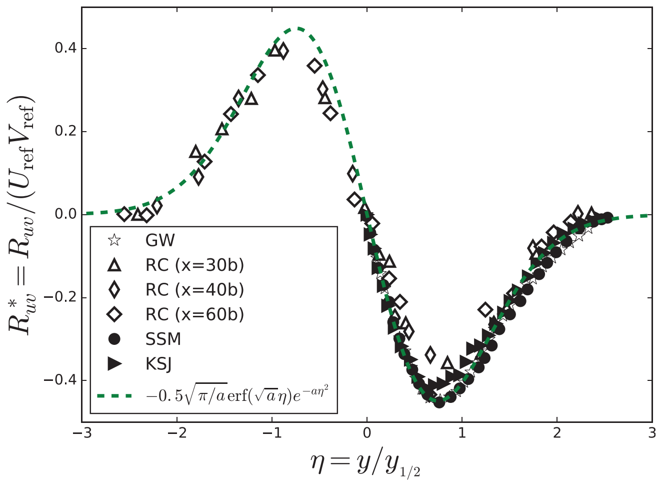

FIG. 4. Scaled Reynolds shear stress $R_{uv}/(-\delta dU_{\rm ctr}/dxU^2_{\rm ctr})$ and the approximate equation [Eq. (30)]. Data Refs. 8, 12, 13, and 28 as in Fig. 2.

or in a dimensional form as

$$\int_{-\infty}^{\infty} U^2 dy \approx 1.5U^2_{\rm ctr}\delta. \tag{32}$$

The integral constraint [Eq. (27)] dictates that $U^2_{\rm ctr}\delta$ is a constant. In other words, the momentum flux in a turbulent plane jet is conserved.[1,7,10]

III. DISCUSSION

In experimental studies of free shear turbulence or a zero-pressure-gradient turbulent boundary layer (ZPG-TBL), it is extremely challenging to obtain accurate measurements of the mean transverse or wall-normal velocity due to its small magnitude. Thus, the proper scaling for the mean transverse or wall-normal flow has not been clear in previous research. A traditional view of wall-bounded turbulence holds that friction velocity $u_\tau = \sqrt{\tau_{\rm w}/\rho}$ is a proper velocity scale in the near-wall region for both the streamwise flow and wall-normal flow. Here, $\tau_{\rm w}$ is the wall shear stress and ρ is the fluid density.[25] However, Wei and Klewicki[38] recently found that the proper scale for the mean wall-normal flow in the ZPG-TBL is not the friction velocity but the mean wall-normal velocity outside the boundary layer V_∞. Moreover, Wei and Maciel[39] proposed a new mixed scale for the Reynolds shear stress in the ZPG-TBL as $R_{uv,\rm ref} = U_\infty V_\infty$, where U_∞ is the free stream velocity. These new findings[38,39] are distinctively different from the traditional view, and the new scaling highlights the critical role of the mean wall-normal velocity in the scaling of the ZPG-TBL.

In previous studies of turbulent plane jets, the jet centerline velocity $U_{\rm ctr}$ has been typically used to scale all the mean flow statistics, including the mean axial flow, the mean transverse flow, and the Reynolds shear stress. In other free shear turbulence, such as turbulent wakes or mixing layers, the mean axial velocity scale is also used to scale the mean transverse flow and Reynolds stresses. If $U_{\rm ctr}$ is used as a scale for the mean transverse flow in a turbulent plane jet, the mean continuity equation [Eq. (7)] becomes

$$0 = \left[\frac{\delta}{U_{\rm ctr}}\frac{dU_{\rm ctr}}{dx} + \frac{d\delta}{dx}\right](U/U_{\rm ctr}) - \left[\frac{d\delta}{dx}\right]\frac{d(\eta\, U/U_{\rm ctr})}{d\eta} + \frac{d(V/U_{\rm ctr})}{d\eta}. \tag{33}$$

The last term in Eq. (33) has a nominal order of magnitude 1, but the nominal orders of magnitude of the other terms are much smaller than 1. Hence, Eq. (33) is not an admissible scaling because it has only one term with a nominal order of magnitude 1. Note that $(U/U_{\rm ctr})$ in the first term or $d(\eta U/U_{\rm ctr})/d\eta$ in the second term of Eq. (33) is $\lesssim O(1)$. Therefore, the last term of Eq. (33) $d(V/U_{\rm ctr})/d\eta$ and $V/U_{\rm ctr}$ will be much smaller than 1.

If $U^2_{\rm ctr}$ is used as a scale for the Reynolds shear stress in a turbulent plane jet, the mean momentum [Eq. (21)] becomes

$$0 = (U/U_{\rm ctr})\frac{d(V/U_{\rm ctr})}{d\eta} - (V/U_{\rm ctr})\frac{d(U/U_{\rm ctr})}{d\eta} + \left[\frac{\nu}{\delta U_{\rm ctr}}\right]\frac{d^2U/U_{\rm ctr}}{d\eta^2} + \frac{d(R_{uv}/U^2_{\rm ctr})}{d\eta}. \tag{34}$$

The viscous term in Eq. (34) is a high order term and does not contribute to the balance of the equation. Note that $(U/U_{\rm ctr})$ in the first term or $d(U/U_{\rm ctr})/d\eta$ in the second term of Eq. (34) is $\lesssim O(1)$, but both $d(V/U_{\rm ctr})/d\eta$ in the first term and $V/U_{\rm ctr}$ in the second term are much smaller than $O(1)$. Therefore, the last term of Eq. (34) $d(R_{uv}/U^2_{\rm ctr})/d\eta$ and $R_{uv}/U^2_{\rm ctr}$ will also be much smaller than $O(1)$. It has been observed that the profiles of mean transverse velocity normalized by $U_{\rm ctr}$, and especially the profiles of the Reynolds shear stress normalized by $U^2_{\rm ctr}$, do not merge to a single curve, and the magnitude of such scaled profiles is indeed much smaller than $O(1)$.

In this paper, we show that a proper velocity scale for the mean transverse flow in a turbulent plane jet is the mean transverse velocity outside the jet $V_{\rm ref} \approx |V_\infty|$, and a proper scale for the Reynolds shear stress is also a mixed scale $R_{uv,\rm ref} = U_{\rm ref}V_{\rm ref} \approx U_{\rm ctr}|V_\infty|$. In fact, the scaling of the mean transverse flow and Reynolds shear stress in turbulent plane jets and ZPG-TBL bears a striking similarity. This similarity is not surprising because the governing equations for the mean flow in the outer layer of ZPG-TBL are identical to those for the turbulent plane jet. However, there are also important differences between the free shear turbulence and wall-bounded turbulence, most notably in terms of boundary conditions. In the ZPG-TBL, the boundary condition for the mean streamwise velocity outside the boundary layer is U_∞. In contrast, in the turbulent plane jet, the mean axial velocity outside the jet is zero. Another important difference is that the width of the turbulent plane jet grows linearly in the axial direction,[1] but the growth of the boundary layer thickness in the streamwise direction is slower than a linear function.[2]

IV. SUMMARY

The scaling patch approach, originally developed for shear-driven wall-bounded turbulence, is applied to determine the proper scales in the turbulent plane jet. A proper velocity scale for the mean transverse flow is found as $V_{\rm ref} = -\delta\, dU_{\rm ctr}/dx$, which is essentially the mean transverse velocity outside the jet $V_{\rm ref} \approx |V_\infty|$. A proper scale for the Reynolds shear stress is found as $R_{uv,\rm ref} = U_{\rm ref}\, V_{\rm ref}$, which is a mix of the velocity scales for the mean axial and transverse flows. The results for the turbulent plane jet are summarized in Table III.

TABLE III. Summary of turbulent plane jet results.

Mean axial velocity scale	$U_{\text{ref}} = U_{\text{ctr}}$
Mean transverse velocity scale	$V_{\text{ref}} = -\delta \dfrac{dU_{\text{ctr}}}{dx}$
Reynolds shear stress scale	$R_{uv,\text{ref}} = U_{\text{ref}} V_{\text{ref}}$
Continuity equation	$0 = U^* - 2\dfrac{d(\eta U^*)}{d\eta} + \dfrac{dV^*}{d\eta}$
Momentum equation	$0 = U^* \dfrac{dV^*}{d\eta} - V^* \dfrac{dU^*}{d\eta} + \dfrac{dR^*_{uv}}{d\eta}$
Equation for V^*	$V^* = 2\eta U^* - \int_0^\eta U^* d\eta$
	$\approx 2\eta e^{-a\eta^2} - \dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}\text{erf}\left(\sqrt{a}\eta\right)$
Equation for R^*_{uv}	$R^*_{uv} = -U^* \int_0^\eta U^* d\eta$
	$\approx -\dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}\text{erf}\left(\sqrt{a}\eta\right)e^{-a\eta^2}$
Volumetric flow rate	$\int_{-\infty}^{\infty} U^* d\eta = 2\lvert V^*_\infty \rvert$
Momentum flow rate	$\int_{-\infty}^{\infty} (U^*)^2 d\eta \approx 1.5$

Applying the new scaling, the scaled mean transverse velocity or Reynolds shear stress profiles at different axial locations merge well onto a single curve, that is, they display self-similarity. An approximate function for the self-similar mean transverse velocity profile is developed as $V^* \approx 2\eta e^{-a\eta^2} - 0.5\sqrt{\pi/a}\,\text{erf}\left(\sqrt{a}\eta\right)$, and an approximation function for the self-similar Reynolds shear stress profile is developed as $R^*_{uv} \approx -0.5\sqrt{\pi/a}\,\text{erf}\left(\sqrt{a}\eta\right)e^{-a\eta^2}$. These approximation functions are shown to agree well with experimental and numerical data, making them useful in assessing the results of future experimental or numerical studies of the turbulent planet jet.

The present analysis demonstrates that the admissible scaling, originally developed for wall-bounded turbulence, can also be applied to free-shear turbulence. In specific, this work reveals the critical role of the mean transverse flow on the scaling and understanding of turbulent plane jets. The similarity and differences of the scaling between the turbulent plane jet and the outer layer of shear-driven wall-bounded turbulence are also discussed. Previous studies of wall-bounded turbulence and wall-free turbulence are, to some degree, separated, and the present work opens a pathway for future work to advance our understanding of these two different turbulences.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created in this study.

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