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Letter to the Editor

Lateral displacement in small angle multiple scattering

The Editor,
Sir,

Values for the average lateral displacement y_{RMS} in small angle multiple scattering of protons with energies of several hundred MeV have recently been published (Mustafa and Jackson 1981, abbreviated as MJ). In the calculations producing these results, an approximation was made with equation (MJ, 50) which leads to systematic errors in the calculation of y_{RMS}. The basic form of this approximation can be understood by examining the discussion provided by Eyges (1948). He derived the general solution for the *gaussian approximation* of both the distribution in projected angle ϕ and for lateral deflections y in his equations (15) and (16)†, implying

$$\phi^2_{RMS} = 2A_0 = \tfrac{1}{2}\int \theta_s^2(x)\,dx \tag{1}$$

$$y^2_{RMS} = 2A_2 = \tfrac{1}{2}\int \theta_s^2(x)(t-x)^2\,dx \tag{2}$$

where $\theta_s(x) = E_s/p\beta$ (equivalent to equation (MJ, 34)), p, β = momentum and velocity of particle at position x, and the integrals extend over the absorber thickness from 0 to t. If t is small compared to the range of the particles, (so that $\theta_s^2(x)$ can be considered constant), the integrals (1) and (2) give the results of Rossi and Greisen (1941), in particular

$$\phi^2_{RMS} = \tfrac{1}{2}\theta_s^2 t = \tfrac{1}{2}\theta^2_{RMS} \tag{3}$$

$$y_{RMS} = \phi_{RMS}t/\sqrt{3} = \theta_{RMS}t/\sqrt{6}. \tag{4}$$

For larger thicknesses, this relation is not valid (even if ϕ_{RMS} is calculated correctly with the integral of equation (1)) and the energy loss must be taken into account as in equation (2). It is readily seen that the weighting factor $(t-x)^2$ will tend to reduce the influence of the larger scattering angles at the smaller energies, giving smaller values of y_{RMS} than would be obtained with equation (4).

Moliere (1955) has derived the distribution of the lateral displacement y without making the gaussian approximation. For an absorber of thickness t it is given by

$$f(t, y) = \exp(-Y^2) + f^{(1)}(Y)/B_y + f^{(2)}(Y)/B_y^2 \tag{5}$$

where $Y = y/y_0$, $y_0 = y_c\sqrt{B_y}$ and y_c and B_y are calculated with integrals similar to those of equations (MJ, 52) and (MJ, 53), but with the weighting factor $(t-x)^2$ suggested by equation (2) above. The 'characteristic displacement' y_0 is equal to $y_{RMS}\sqrt{2}$ of the

† The following corrections should be made: equation (14), change $4A_0B$ to $4B$, equation (15): change $4A_0t$ to $4A_0$.

gaussian term of equation (5). We have written a computer program based on the Moliere expressions for the evaluation of y_0 and have included the calculation of χ_c and B for the angular deflection (as given in equations (MJ, 52) and (MJ, 53)) for any desired energy loss in any material (with the possibility of arranging different substances of arbitrary composition in several parallel layers, and also including gaps).

In order to compare our results with those of MJ, we note that it would be preferrable to denote the angle given in equation (MJ, 43) with the symbol $\theta_0 = \chi_c\sqrt{B}$, calling it the 'characteristic angle' of the Moliere distribution $f(t, \theta)$. Similarly, the quantity ${}_a y_0$ shall be defined by

$$ {}_a y_0 = \theta_0 t/\sqrt{3}. \tag{6}$$

The values of y_{RMS} given by MJ in table 5 are equal to ${}_a y_0/\sqrt{2}$.

A comparison of y_0 and ${}_a y_0$ calculated with our program is given in table 1 for protons with an initial energy of 350 MeV in water. The final energy, E_1, as well as the absorber thickness t are given. The values for ${}_a y_0$ differ slightly from those given by MJ because different parameters were used in the calculations (in particular, $I = 75$ eV was used). As anticipated, y_0 is considerably smaller than ${}_a y_0$ for large energy losses. This is caused by the weighting factor $(t-x)^2$ in the Moliere integrals for y_c and B_c. Since the spread in residual energies will be very large for $E_1 < 40$ MeV, the values of y_0 and ${}_a y_0$ should be considered indicative only for these energies.

Table 1. Comparison of the average lateral displacement for protons with an initial energy of 350 MeV in water, y_0 calculated with Moliere's theory, ${}_a y_0$ with the approximation described by equation (4). The quantity y_{RMS} given by MJ is ${}_a y_0/\sqrt{2}$.

E_1 (MeV)	t (mm)	y_0 (mm)	${}_a y_0$ (mm)	Difference (%)
320	90.5	0.79	0.82	3.5
280	204	2.82	3.01	6.6
240	309	5.48	6.06	10
200	403	8.51	9.83	15.5
160	486	11.7	14.3	22
120	556	14.8	19.5	31
80	611	17.6	25.6	46
60	632	18.8	29.4	57
40	648	19.7	34.0	72
30	654	20.1	36.9	—
20	659	20.4	40.7	—

It should be noted that for absorber thicknesses exceeding a few centimetres, the distributions $f(t, \theta)$ and $f(t, y)$ function may change drastically at values below about 10% of the maximum values $f(t, 0)$ due to the contribution of nuclear scattering. Furthermore, the influence of the term with B^3 in the Moliere expansion of the distribution function should be examined if $B < 10$.

The 'effective gaussian core' of the Moliere distribution may be characterised by $y_{1/e} = y_0(1.011 - 0.87/B_y)^{1/2}$ and its width is not equal to y_0 owing to the influence

of the second and third term in equation (5). Also, it should be pointed out that y_{RMS} does not exist in principle since $f^{(1)}(Y)$ behaves as Y^{-1} for large values.

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Note added in proof

Mustafa and Jackson (whose work is referenced in the above letter) would like to make the point that while the argument given by Bichsel is correct, it can be seen from the table that the error in their calculations of lateral displacement is at most 10%.