LIMITED ANGLE CT RECONSTRUCTION USING A PRIORI INFORMATION*

Kenneth M. Hanson
Group M-4

Los Alamos National Laboratory
Los Alamos, New Mexico

Abstract

Projection data that are limited in number and range of viewing angle cannot completely specify an arbitrary source function. In the space of all permissible functions there exists a null subspace about which the projection measurements provide no information. Deterministic reconstruction algorithms usually set the null space contributions to zero leading to severe reconstruction artifacts. A Fit And Iterative Reconstruction (FAIR) method is proposed that incorporates a priori knowledge of the approximate functional form of the source. In FAIR the parameters of this functional model are determined from the available projection data by a weighted fitting procedure. The resulting distribution is then iteratively revised to bring the final estimate into agreement with the measured projections using a standard algorithm such as ART.

Introduction

There are many situations in which it would be desirable to obtain decent tomographic reconstructions of an object from projection data that are limited in number and/or range of viewing angle [1]. Unfortunately there are severe limitations imposed upon deterministic reconstruction algorithms by limited angle projection data that cannot be overcome without the use of a priori knowledge about the object to be reconstructed [1]. We will review these limitations that arise from the null space corresponding to the available projection data. A new approach that incorporates the expected shape of the reconstructed object will be shown to circumvent the difficulties encountered by deterministic methods.

Measurement space - null space

The CT problem may be stated as follows: given a finite set of projections of a function of two dimensions f(x,y) with compact support, obtain the best estimate of that function. The projections may generally be written as a weighted 2-D integral of f(x,y)

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$$p_i = \iint h_i(x,y)f(x.y)dxdy, \qquad (1)$$

where the h_i are the weighting functions and i= 1, 2, ...N for N individual measurements. We will refer to the h_i as response functions. In the CT problem the h_i typically have large values within a narrow strip and small or zero values outside the strip. If the h_i are unity within a strip and zero outside, eq. 1 becomes a strip integral. For zero strip width, it becomes a line integral. These latter two cases are recognized as idealizations of the usual physical situation. The generality of eq. 1 allows it to closely represent actual physical measurements since it can take into account response functions that vary with position.

The unknown function f(x,y) is usually restricted to a certain class, e.g., the class of all integrable functions with compact support. Consider the space of all acceptable functions and assume that all the h_i belong to that space. Then eq. 1 has the form of an inner product of h_i with f. That is, p_i may be thought of as a projection of the unknown vector f onto the basis vector h_i. Only those components of f that lie in the subspace spanned by the set of all h_i contribute to the measurements. We will call this subspace the measurement space. The components of f in the remaining orthogonal subspace, the null space, do not contribute to the measurements. Hence, the null space contribution to f cannot be determined from the measurements alone. Since the deterministic (measurement) subspace of f is spanned by the response functions, it is natural to expand the estimate of f in terms of them

$$\hat{f}(x,y) = \sum_{i=1}^{N} a_i h_i(x,y). \qquad (2)$$

This is equivalent to setting the null space components of f to zero, which yields the minimum norm solution. This leads to artifacts in $\hat{f}$ since it does not possess those components of f that lie in the null space. Further reading on the null space - measurement space concept may be found in papers by Twomey [2,3,4].

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The response function expansion, eq. 2, is formally identical to the familiar backprojection process where the value a_i is added to the image along the strip function h_i. Thus, the backprojection process only affects the measurement space components of the reconstruction. Most of the well-known CT reconstruction algorithms incorporate backprojection including filtered backprojection[5], ART[6], SIRT[7], SIRT-like algorithms (least squares[8] and other variants[9], and the "natural" pixel matrix formulation by Buonocore et al.[10,11]. Such algorithms can only alter the measurement space part of the initial estimate. When the initial estimate lies solely in the measurement space, as is normally the case, so will the final estimate.

The effect of the restriction of deterministic solutions to the measurement space may be demonstrated by means of an example. Consider Fig. 1a to be an object to be reconstructed. Suppose that eleven parallel projections are taken of the object within a 90° angular range. Each projection contains 128 samples and is subject to a slight degradation in spatial resolution. The null space contribution to the original image may be readily calculated by using Fig. 1a as the initial estimate for an iterative reconstruction algorithm and setting the input projections to zero. In this example an ART algorithm was used. The algorithm alters the measurement space part of the initial image (by means of backprojection) until the result has zero projection values in the measured projection directions. The result, Fig. 1b, is the null space part of the original image. Subtraction of Fig. 1b from Fig. 1a yields the measurement space part of the original image, Fig. 1c. The null space component, Fig. 1b, is the part of the object that cannot be determined from the measurements alone. The sections of the annulus that are roughly tangential to the response functions are nearly zero except for the abrupt changes at the edge due to finite spatial resolution. The upper-left and lower-right portions of the annulus are determined the worst. The measurement space part of the object, Fig. 1c, is the minimum norm solution consistent with the measurements and is the best that can be expected from any linear, deterministic algorithm.

Various augmentations to deterministic algorithms such as consistency, analytic continuation, and global constraints (including maximum entropy) have been considered by Hanson[1]. These seem to be ineffective in overcoming the measurement space restrictions presented above. Other authors have mentioned in passing the concept of the measurement space-null space dichotomy[12,13,14,15] but have not considered its effect on reconstructions from limited projection data. As an aside, the range of the transpose of the projection measurement matrix A referred to in Ref. 15 is the measurement space in the square pixel representation. Louis[16] has shown that spurious ghosts can arise from the null space corresponding to a finite set of projection data. Further references on the limited angle CT problem may be found in Ref. 1.

The restriction of deterministic solutions to the measurement space should not be viewed as a negative conclusion. Rather it is simply a statement of what is possible for a given set of measurements in the absense of further information. It allows one to formally state the goal in limited angle CT reconstruction as that of estimating the null space contribution through the use of further information about the function to be reconstructed.

FAIR - use of a priori knowledge

We have seen in the foregoing development that deterministic solutions are deficient because of their lack of a null space contribution. Thus, we are led to supplement the available measurements with additional information about the object to be reconstructed in order to obtain some reasonable estimate of its null space component. A priori knowledge may take many forms. For example, it may be known that the values of the function to be reconstructed are restricted in any of several ways such as upper and/or lower limits or known discrete values. A commonly used reconstruction constraint is that of positivity (strictly, nonnegativity) since the quantities often being reconstructed, linear attenuation coefficients or isotope densities, are known not to have negative values. Positivity can exert a strong influence on the reconstruction result in cases where the reconstruction should be zero in a large portion of the reconstruction region. In other situations it may be useless. Another type of a priori knowledge it might be that it is known that the object to be reconstructed is taken from a well-defined ensemble of objects. Then the reconstruction procedure could be based upon the ensemble probability distributions as in maximum a posteriori probability reconstruction using a SIRT-like algorithm[9] or as in a Karhunen-Loève expansion[17]. This approach may prove to work well only in situations where the ensemble statistics are sufficiently restrictive.

We wish to introduce a new method for using a priori knowledge about the shape or form of the object to be reconstructed. In this two step approach, the fit and iterative reconstruction (FAIR) technique, it is assumed that a parametric model roughly approximating the object can be specified. The first step is to fit the model parameters in a least square (or minimum chi squared) sense to the available projection data. The second step is to employ an iterative reconstruction algorithm, such as ART, using the fitted model as the initial estimate. This step is needed since the functional model used in the first step may be necessarily crude and its projections may not fully agree with the measurements. As discussed earlier, the second step only affects the measurement space part of the reconstruction bringing it into agreement with the available projection data. The first step may be viewed as providing a reasonable guess for the null

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space contribution consistent with the functional model. This approach will be demonstrated below by means of two examples.

An important advantage of using the fitting procedure in the FAIR technique is its flexibility. Additional parameters may be employed to allow the position, orientation and size of the object to be adjusted. It is also possible to incorporate constraints on the parameters to avoid unrealistic objects. For example, in the annulus problems below, it would be possible to allow each of the 2-D gaussian basis functions to be centered at an arbitrary radius instead of at a fixed radius. This would permit the size and contour of the reconstructed annulus to be determined from the measurement data. The reconstruction could still be restricted to an annular shape by adding a penalty function to chi squared based on the quadratic difference between the radii of adjacent gaussians. This would tend to force the radii to be a smooth function of polar angle.

The iterative reconstruction algorithm used here in the second step of FAIR is a version of ART developed at Los Alamos. In this version, the present estimate of the reconstruction is stored as a square pixel representation. The usual technique is used of backprojecting differences between projections of the present estimate and the input projections. The algorithm used here differs from earlier versions of ART[9] in that the basic projection and backprojection computations are carried out in a way to more accurately represent the corresponding analytic processes than the simpler nearest neighbor assignment of pixels to projection rays. The present ART routine typically converges in three to five iterations to a stable solution that does not change appreciably in subsequent iterations (up to 20). When faced with noisy input projections, the reconstruction tends to diverge slowly after five iterations as has been observed before for ART[18].

Example 1

The first example will be based on the object shown in Fig. 1a that resembles a thick-walled pipe with two inclusions. Where the image width is 1.0, the inner and outer radii of the annulus are 0.2 and 0.3, respectively. The two small circles have a diameter of 0.05 and are at half the density of the annulus. It will be assumed that eleven parallel, noiseless projections over a range of 90° are available. Each projection consists of 128 samples. The unconstrained ART reconstruction starting with the average value, Fig. 2a, is virtually identical with the measurement space component of the object, Fig. 1c. The positivity constraint greatly reduces the streaking artifacts, Fig. 2b, but does not eliminate the squaring off of the rear and far sides of the annulus that arises from the lack of projections over the remaining 90°. The maximum entropy algorithm MENT[19] produces a very similar result, Fig. 2c. MENT does not degrade the spatial resolution as much as ART because the representation of the MENT reconstruction is directly related to the response function expansion instead of the square pixel representation used in ART. Besides distorting the shape of the annulus, all of these reconstructions make it difficult to observe the two small circles.

It will be assumed that it is known a priori that the object to be reconstructed has an annular shape of known radius and width. Let us choose for a model of this object a linear combination of 18 two dimensional gaussian distributions whose centers are equally spaced on a circle of appropriate radius. The FWHM of the gaussians is the same as the width of the annulus. The amplitudes are to be determined by fitting the projections of this functional model to the projection data. It is realized at the outset that this model is a crude representation of the actual object but it will yield a distribution restricted to an annulus and has the computational advantage that its projections are easily calculated. Fig. 3a shows the result of fitting the amplitudes of the 18 gaussians to best match the 11 projections, the first step in FAIR. There is hardly a hint of the two small circles since the amplitudes have been severely distorted to make up for the discrepancy between the assumed and actual cross sections. However, using Fig. 3a as the starting distribution, the ART algorithm produces the final results, Fig. 3b without positivity and Fig. 3c with positivity. These results provide much better visualization of the small circles in the original object than the reconstructions in Fig. 2. The major advantage of starting ART with Fig. 3a is that Fig. 3a properly positions the near and far sections of the annulus and thus the squaring off is avoided. The incorporation of a reasonable estimate of the null space contribution through Fig. 3a is seen to greatly improve the reconstruction result.

It has been proposed[20,21,22,23] that one way to overcome the limitations arising from limited projection angles is to exploit a priori information concerning the region of support of the unknown function. One way to do this is to use an adaptation of the Papoulis-Gerchberg technique in which the known properties of the function are alternatively enforced in the spatial and Fourier domains. It has been shown[21,22] that this technique is not sufficient to recover all the degrees of freedom in the original function. The ART reconstruction algorithm may be easily altered to incorporate a known region of support and upon convergence the result should be identical to that obtained by the above technique. The result of specifying a circular region of support just outside the annulus in Fig. 1a is shown in Fig. 4a. This certainly improves the reconstruction (compared with Fig. 2b) but does not reproduce the original image.

We have seen in the FAIR results that the initial choice for ART can make a big difference in the final result. Fig. 4b shows the ART reconstruction that results when a flat annulus of proper dimensions is used for the starting distribution. This would be a reasonable guess if it were known that the object being examined was a pipe

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but the presence of the small holes was unknown. Figure 4b is a very good reconstruction because the null space part of Fig. 1a is well specified by the flat annulus. However, if a flat annulus of the wrong size is chosen for the starting distribution, the result (Fig. 4c) is much worse than the standard ART result, Fig 2a. It is important that the starting distribution be representative of the unknown object. The advantage of the FAIR approach is evident since the parameters for the flat annulus model could easily and accurately be obtained from the projection data to yield Fig. 4b.

It is important to realize that all of the ART reconstructions shown in this section have the same projections at the measured projection angles, i.e. their measurement space contributions are identical. They differ only in their null space contributions, these being determined by the combined effect of constraints and starting distributions. This observation indicates the enormity of the ambiguity present in the measurements that can only be reduced by use of a priori information.

Example 2

The second example will be based on an annulus with gaussian cross section and variable amplitude, Fig. 5a. This object is similar to the blurred cross section of the Thallium 201 distribution taken up in heart muscle. The straightforward ART reconstructions from 11 viewings subtending 90°, Fig. 5b and c, show the same types of artifacts as in the preceeding example. The hole in the upper-right quadrant has virtually disappeared while that in the lower right has been greatly exaggerated. Use of the 18-gaussian annulus model described above in the fitting step of FAIR yields, Fig. 6, a decent representation of the original object because the gaussian basis functions provide a good approximation to the gaussian cross section of the annulus. When this is used as the starting distribution in an unconstrained ART algorithm, the final result, Fig. 6b, reproduces the original distribution very well. Since this starting distribution so closely matches the projection data, the positivity constraint has little effect on the result. Figure 7 shows that, even when an annulus with gaussian cross section and constant amplitude is used for the starting distribution, the unconstrained ART reconstruction yields an acceptable result. This indicates that reconstruction methods based on ensemble statistics may work well since their initial estimate is the ensemble mean, which could be a flat annulus for Thallium 201 distributions in the heart walls.

The foregoing results are quantitatively summarized in Fig. 8. The maximum reconstruction value between radii of 0.5 and 1.5 times the mean annulus radius is plotted versus polar angle. It is observed that the FAIR result that starts with the fitted distribution, Fig. 6a, comes remarkably close to the original distribution. This is contrasted by the conventional ART reconstruction that starts with the average value which does very poorly. Again, the use of a priori knowledge to estimate the null space contribution is very beneficial.

Discussion

It has been shown that artifacts arising in the limited angle CT problem can be reduced by properly estimating the null space contributions of the unknown function. This can only be accomplished through the use of a priori knowledge concerning the source function. In the FAIR technique presented here, the initial estimate of the function is obtained by fitting the parameters in a functional model of the object to the available projection measurements. The null space contribution of this estimate survives the subsequent iterative reconstruction procedure to reduce the artifacts in the result. As with most image processing schemes, this new technique must be tried in each new imaging problem to assess its worth since experience is not easily transferred. We have attempted to provide the reader with some understanding of the behavior of the FAIR technique by means of several examples.

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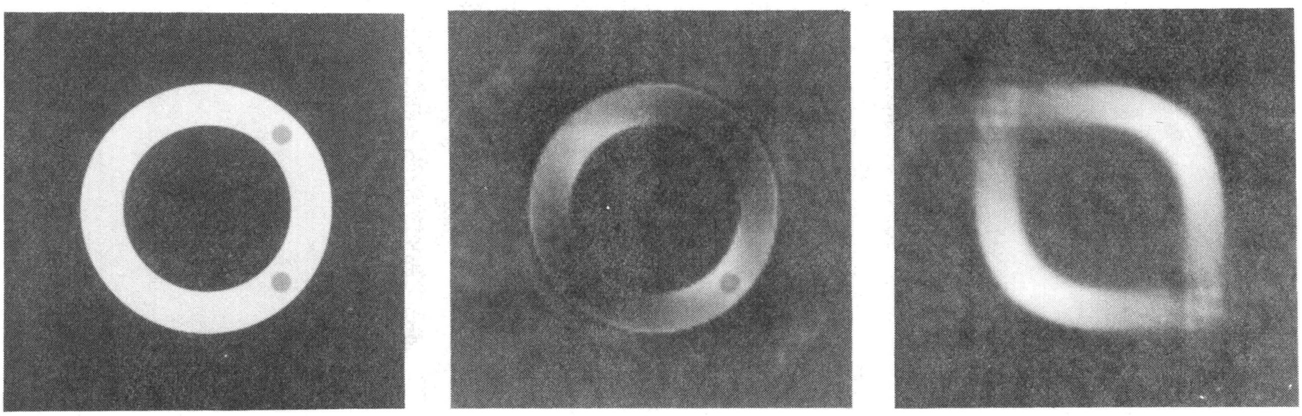
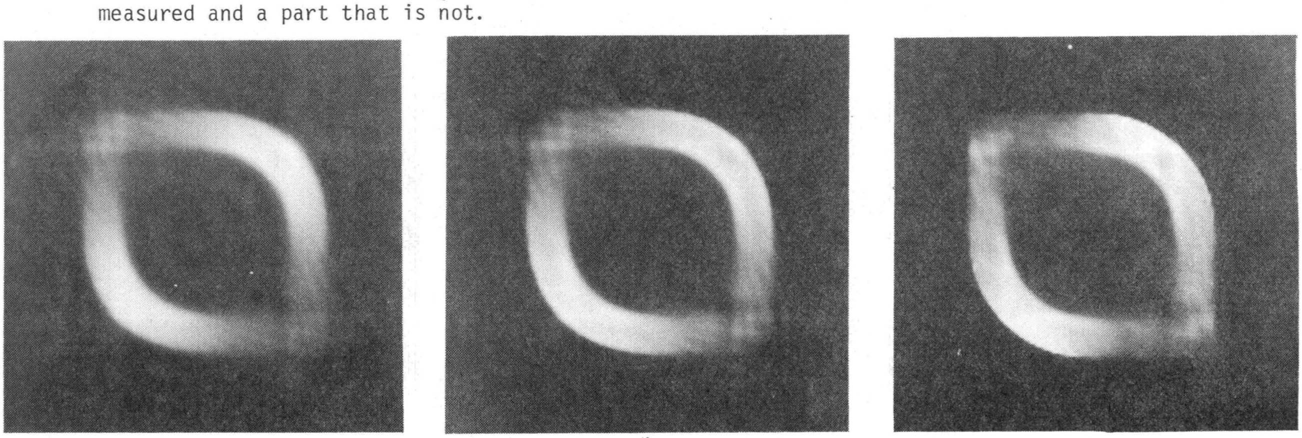


a b c

Fig. 1 - The decomposition of an object consisting of an annulus with two holes a) into its measurement space b) and null space c) contributions corresponding to 11 measured projections covering 90°. This illustrates that for a given measurement scheme, any function is the sum a part that is measured and a part that is not.



a b c

Fig. 2 - Reconstructions of Fig. 1a from 11 views covering 90° using a) unconstrained ART (average starting value), b) ART with positivity, and c) the maximum entropy algorithm MENT.

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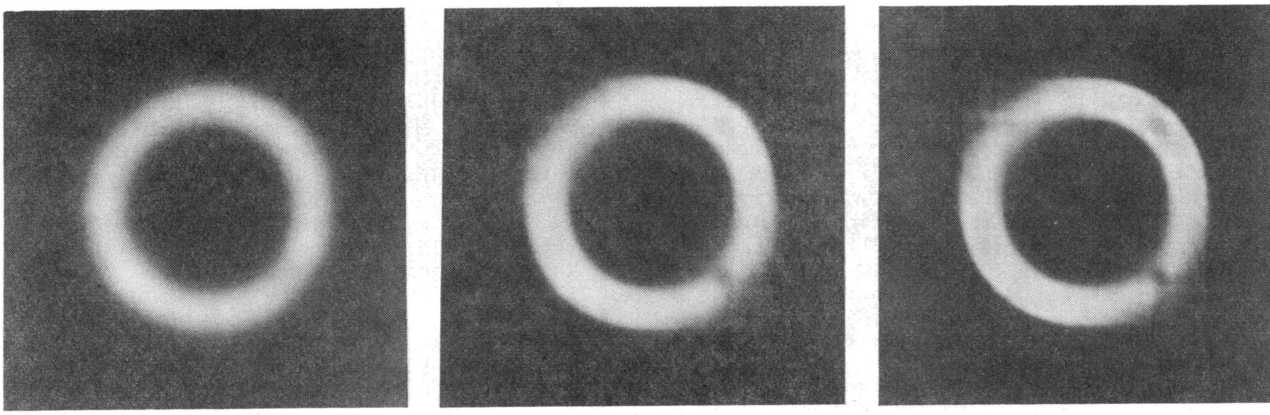
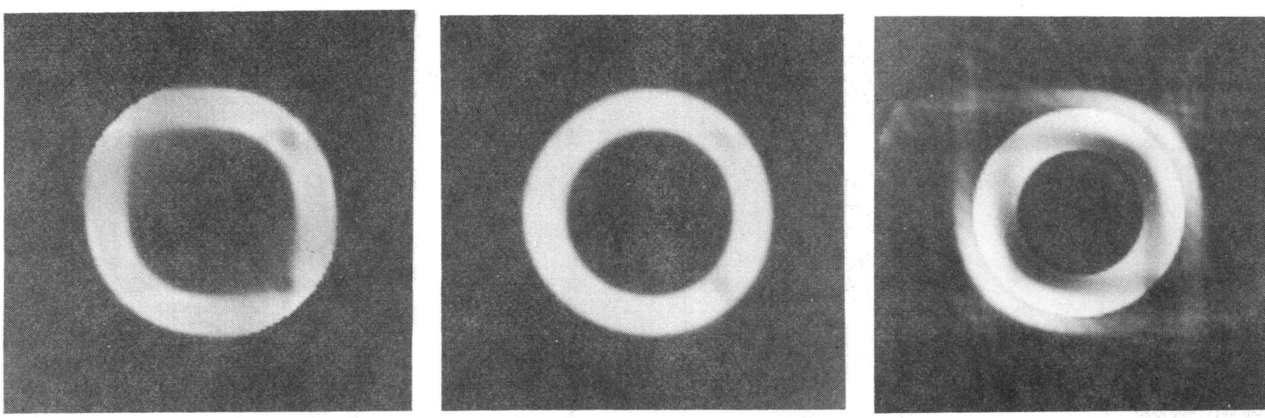


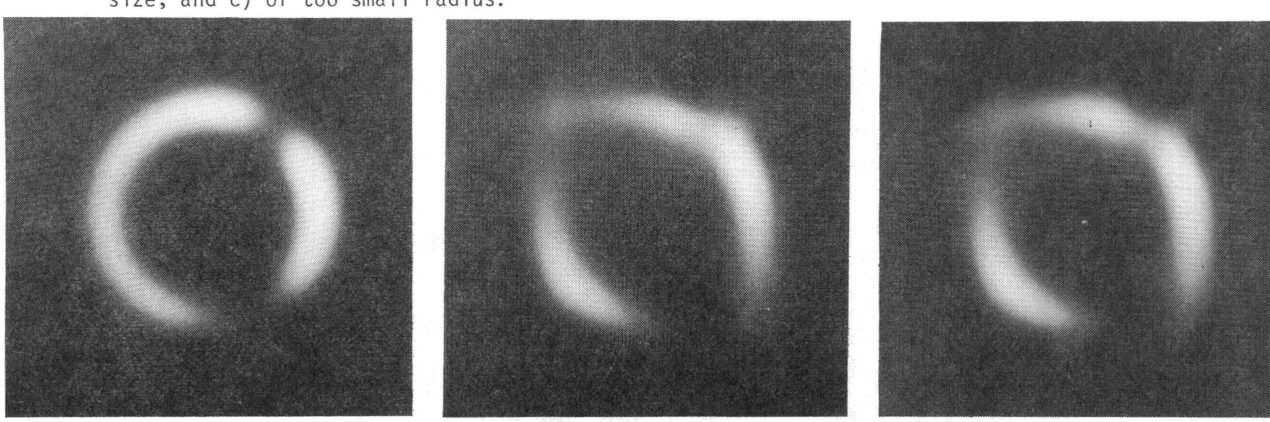
a b c

Fig. 3 - The Fit And Iterative Reconstruction (FAIR) results for Fig. 1a from 11 views covering 90° showing a) the 18-gaussian fit to the measurements used for the initial guess in the subsequent ART reconstruction, b) without positivity, and c) with positivity.



a b c

Fig. 4 - Reconstruction of Fig. 1a from 11 views covering 90° using ART a) with positivity and the reconstruction region limited to a circle that is slightly larger than the annulus. Reconstruction from same projections with a uniform annulus used for the starting distribution, b) of proper size, and c) of too small radius.



a b c

Fig. 5 a) An annulus with gaussian cross section and variable amplitude and its reconstruction from 11 views covering 90° using b) unconstrained ART (average starting value) and c) ART with positivity constraint.

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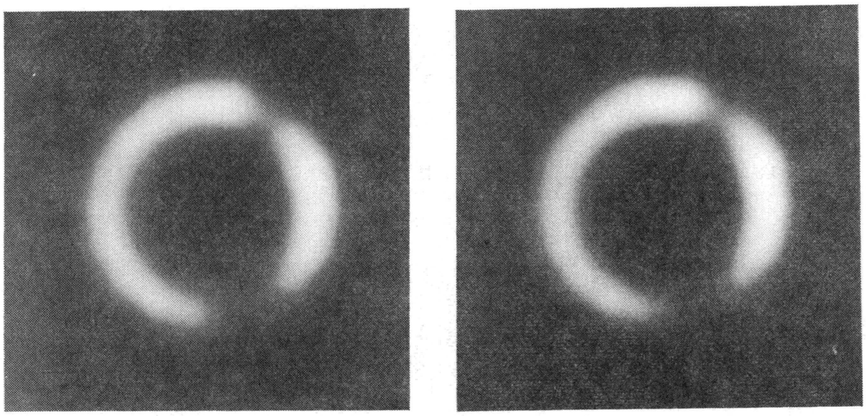

a b

Fig. 6 - FAIR reconstruction of Fig. 5a from 11 views covering 90° showing the a) 18 gaussian fit and final ART reconstruction b) without positivity constraint. The use of a positivity constraint makes little difference.

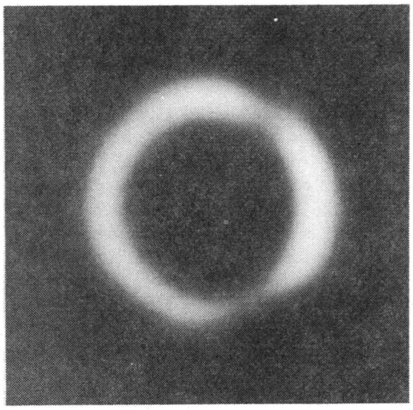

Fig. 7 - Reconstruction of Fig. 5a from 11 views covering 90° using an initial guess of an annulus with gaussian cross section and constant amplitude in ART algorithm.

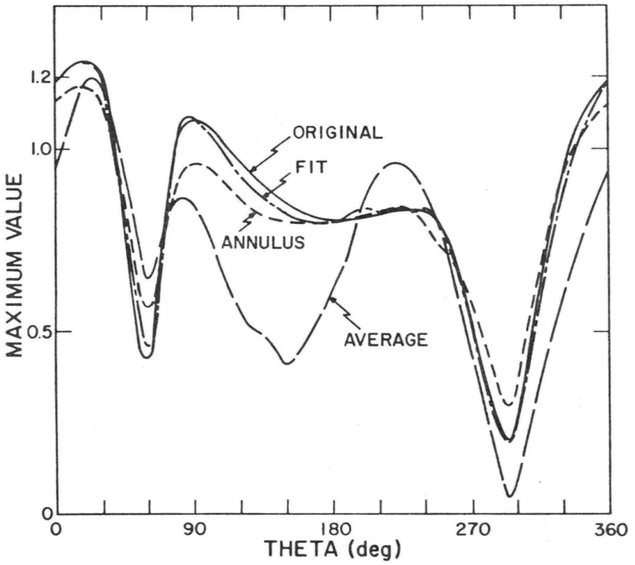

Fig. 8 - Angular dependence of maximum reconstruction value between radii of 0.5 to 1.5 times mean annulus radius for positivity constrained ART reconstructions employing various starting distributions. The fitted starting distribution used in FAIR (Fig. 6b) closely matches the original object while starting with the constant amplitude annulus (Fig. 7) does remarkably well. The conventional initial guess of a constant distribution with correct value (Fig. 5c) results in poor agreement.

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