BLOCK-DIAGONAL REPRESENTATIONS FOR COVARIANCE-BASED ANOMALOUS CHANGE DETECTORS

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ABSTRACT

We use singular vectors of the whitened cross-covariance matrix of two hyper-spectral images and the Golub-Kahan permutations in order to obtain equivalent tridiagonal representations of the coefficient matrices for a family of covariance-based quadratic Anomalous Change Detection (ACD) algorithms. Due to the nature of the problem these tridiagonal matrices have block-diagonal structure, which we exploit to derive analytical expressions for the eigenvalues of the coefficient matrices in terms of the singular values of the whitened cross-covariance matrix. The block-diagonal structure of the matrices of the RX, Chronochrome, symmetrized Chronochrome, Whitened Total Least Squares, Hyperbolic and Subpixel Hyperbolic Anomalous change detectors are revealed by the white singular value decomposition and Golub-Kahan transformations. Similarities and differences in the properties of these change detectors are illuminated by their eigenvalue spectra.

***Index Terms*—** change detection, anomalous change detection, hyper-spectral, eigenvalues, tridiagonal matrix, block-diagonal matrix

1. INTRODUCTION

Anomalous Change Detection (ACD) methods aim to identify rare, unusual or anomalous changes [1] and are of crucial importance in many remote sensing applications, such as monitoring and surveillance. A number of ACD algorithms can be expressed as quadratic functions of the data, where the coefficients are based on the covariances and cross-covariances of two images [2] being compared. Among these methods are the RX [3], Chronochrome [4], Whitened Total Least Squares (WTLSQ) [5], Covariance Equalization [6], Multivariate Alteration Detection [7], Hyperbolic [8] and Subpixel Hyperbolic [9] methods. The eigenvalue spectrum of coefficient matrices can provide valuable insights into the algebraic and numerical properties of the covariance-based quadratic ACD methods.

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2. ANOMALOUS CHANGE DETECTORS

Consider two hyper-spectral images $D_x = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$ and $D_y = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_N)$ where $\mathbf{x}_i \in \mathbb{R}^{d_x}$ and $\mathbf{y}_i \in \mathbb{R}^{d_y}$ are corresponding pixels in the same scene. We assume, without loss of generality, that the pixels in the images D_x and D_y have zero mean. The scalar measure of anomalousness, when comparing pixels $\mathbf{x}_i$ and $\mathbf{y}_i$, is, for a large class of ACD algorithms, given by [2]

$$\mathcal{A}(\mathbf{x}_i, \mathbf{y}_i) = (\mathbf{x}_i^T \; \mathbf{y}_i^T)\, Q \begin{pmatrix} \mathbf{x}_i \\ \mathbf{y}_i \end{pmatrix}, \tag{1}$$

where the specific form of the quadratic coefficient matrix $Q \in \mathbb{R}^{(d_x+d_y)\times(d_x+d_y)}$ depends on which ACD method is used. The change between the pixels $\mathbf{x}_i$ and $\mathbf{y}_i$ is considered anomalous if $\mathcal{A}(\mathbf{x}_i, \mathbf{y}_i)$ exceeds a given threshold. Here, Q is a dense symmetric matrix that is a function of the covariance and cross-covariance matrices of the two images D_x and D_y:

$$X = \frac{1}{N} D_x D_x^T, \;\; Y = \frac{1}{N} D_y D_y^T, \;\; C = \frac{1}{N} D_y D_x^T. \tag{2}$$

Covariance matrices X and Y are symmetric matrices of size $d_x \times d_x$ and $d_y \times d_y$ respectively, and the cross-covariance matrix C is a rectangular $d_y \times d_x$ matrix.

2.1. Whitened and white SVD coordinates

In the whitened coordinates $\bar{D}_x = X^{-1/2} D_x$, $\bar{D}_y = Y^{-1/2} D_y$, that are used to "normalize" the images with respect to illumination, environmental and other ubiquitous changes [1], the covariance and the cross covariance matrices take the following form $\bar{X} = \bar{D}_x \bar{D}_x^T / N = I$, $\bar{Y} = \bar{D}_y \bar{D}_y^T / N = I$, $\bar{C} = \bar{D}_y \bar{D}_x^T / N = Y^{-1/2} C X^{-1/2}$. Consider the singular value decomposition (SVD) of the whitened cross-covariance matrix $\bar{C} = U \bar{\Sigma} V^T$, where U and V are orthogonal matrices of size $d_y \times d_y$ and $d_x \times d_x$ respectively and $\bar{\Sigma}$ is a rectangular $d_y \times d_x$ matrix

$$\bar{\Sigma} = \begin{cases} \begin{pmatrix} \Sigma \\ 0_{m\times n} \end{pmatrix} & \text{for } d_x < d_y \\[2ex] \begin{pmatrix} 0_{n\times m} & \Sigma \end{pmatrix} & \text{for } d_y < d_x \end{cases} \tag{3}$$

with $n = \min\{d_y, d_x\}$, $m = |d_x - d_y|$, and the diagonal block $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ comprised of singular values $1 \geq \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. To see that the singular values of $\bar{C}$ are less than one, we use the consistency property of the spectral matrix norm [10]:

$$\begin{aligned} \|\bar{C}\|_2 &= \|(D_y\, D_y^T)^{-1/2}\,(D_y\, D_x^T)\,(D_x\, D_x^T)^{-1/2}\|_2 \\ &\leq \|(D_y\, D_y^T)^{-1/2}\, D_y\|_2\, \|D_x^T\,(D_x\, D_x^T)^{-1/2}\|_2 \end{aligned} \tag{4}$$

and singular value decomposition to express $D_y = U_y\,\Sigma_y\,V_y^T$ with orthogonal matrices $U_y \in R^{d_y \times d_y}$ and $V_y \in R^{N \times N}$, and rectangular matrix $\Sigma_y \in R^{d_y \times N}$ consisting of a diagonal block of singular values and a zero block. Then

$$(D_y\, D_y^T)^{-1/2}\, D_y = U_y\,(\Sigma_y\,\Sigma_y^T)^{-1/2}\,\Sigma_y\, V_y^T \tag{5}$$

and $(\Sigma_y\,\Sigma_y^T)^{-1/2}\,\Sigma_y$ is a rectangular matrix with a diagonal block of ones and a zero block. Thus, $\|(D_y\, D_y^T)^{-1/2}\, D_y\|_2 = 1$, and similarly for x, and the inequality in (4) becomes $\|\bar{C}\|_2 \leq 1$. It follows immediately that $\sigma_1 = \|\bar{C}\|_2 \leq 1$.

Similar to the approach used in the Optimal Covariance Equalization [6] and the Diagonalized Covariance Equalization methods [2] we can introduce *white SVD transformation*

$$\begin{aligned} \widetilde{D}_x &= V^T\,\bar{D}_x = V^T\, X^{-1/2}\, D_x, \\ \widetilde{D}_y &= U^T\,\bar{D}_y = U^T\, Y^{-1/2}\, D_y. \end{aligned} \tag{6}$$

It is easy to see that $\widetilde{X} = \widetilde{D}_x \widetilde{D}_x^T / N = I$, $\widetilde{Y} = I$, and

$$\widetilde{C} = (U^T\, Y^{-1/2}\, D_y\, {D_x}^T\, X^{-1/2}\, V)/N = \bar{\Sigma}. \tag{7}$$

We can now define an auxiliary transformation matrix

$$W = \begin{pmatrix} X^{1/2}\, V & 0 \\ 0 & Y^{1/2}\, U \end{pmatrix}, \tag{8}$$

that we will call *white SVD transformation* matrix.

2.2. Tridiagonal and block diagonal structure

Consider the coefficient matrix for the RX method [3, 2]

$$Q_{\text{RX}} = \begin{pmatrix} X & C^T \\ C & Y \end{pmatrix}^{-1}. \tag{9}$$

Applying transformation W to Q_{RX} we obtain white SVD-transformed version of the RX matrix:

$$\widetilde{Q}_{\text{RX}} \equiv W^T\, Q_{\text{RX}}\, W = \begin{pmatrix} I_{d_x} & \bar{\Sigma}^T \\ \bar{\Sigma} & I_{d_y} \end{pmatrix}^{-1}. \tag{10}$$

Notice that

$$\widetilde{Q}_{\text{RX}} = \begin{cases} \begin{pmatrix} I_n & \Sigma & 0 \\ \Sigma & I_n & 0 \\ 0 & 0 & I_m \end{pmatrix}^{-1} & \text{for } d_x < d_y \\[2ex] \begin{pmatrix} I_m & 0 & 0 \\ 0 & I_n & \Sigma \\ 0 & \Sigma & I_n \end{pmatrix}^{-1} & \text{for } d_y < d_x \end{cases} \tag{11}$$

where I_n is an identity matrix of size n.

There always exists an orthogonal permutation Π such that

$$T = \Pi \begin{pmatrix} I & \Sigma \\ \Sigma & I \end{pmatrix} \Pi^T = \overbrace{\Pi \underbrace{\begin{pmatrix} 0 & \Sigma \\ \Sigma & 0 \end{pmatrix}}_{J} \Pi^T}^{G} + I \tag{12}$$

where J is the Jordan-Wielandt matrix [11] and

$$G = \begin{pmatrix} 0 & \sigma_1 & & & & & & \\ \sigma_1 & 0 & 0 & & & & & \\ & 0 & 0 & \sigma_2 & & & & \\ & & \sigma_2 & 0 & 0 & & & \\ & & & 0 & 0 & \ddots & & \\ & & & & \ddots & \ddots & 0 & \\ & & & & & 0 & 0 & \sigma_n \\ & & & & & & \sigma_n & 0 \end{pmatrix} \tag{13}$$

is its permuted tridiagonal form also known as the Golub-Kahan form [11, 12]. This means that $T = G + I$ is a symmetric tridiagonal matrix with ones on the main diagonal and σ_i values interlaced with zeros on the first upper and lower diagonals; that is, matrix T is block-diagonal with each block $i = 1, 2, \ldots, n$ of the form

$$T_i = \begin{pmatrix} 1 & \sigma_i \\ \sigma_i & 1 \end{pmatrix}. \tag{14}$$

Notice that if the matrix $W_\pi = W^T\,\Pi$ is non-singular, transformation $W_\pi\, Q\, W_\pi^T$ of a self-adjoint matrix Q, such as the symmetric matrix of an ACD method (1), is congruent. According to the Sylvester's inertia theorem [13] this transformation preserves inertia of the matrix Q, which is the number of its positive, zero-valued and negative eigenvalues.

We can now define the permuted RX matrix

$$\widehat{Q}_{\text{RX}} = \Pi\,\widetilde{Q}_{\text{RX}}\,\Pi^T = \begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} T^{-1} & 0 \\ 0 & I \end{pmatrix}, \tag{15}$$

where Π is the Golub-Kahan permutation (12). $\widehat{Q}_{\text{RX}}$ is block-diagonal with its first n blocks of the form

$$T_i^{-1} = \frac{1}{1 - \sigma_i^2} \begin{pmatrix} 1 & -\sigma_i \\ -\sigma_i & 1 \end{pmatrix} \tag{16}$$

followed by an identity block. The two eigenvalues of T_i^{-1} in (16) can be solved for directly; they are $1/(1-\sigma_i)$ and $1/(1+\sigma_i)$. Since $\widehat{Q}_{\text{RX}}$ is a block-diagonal matrix, its eigenvalue spectrum $\{\lambda_1^{(RX)}, \ldots, \lambda_{d_x+d_y}^{(RX)}\}$ is the union of the eigenvalue spectra of its blocks.

The Golub-Kahan permutation Π enables the white SVD-transformed inverse coefficient matrix $\widetilde{Q}_{\text{RX}}^{-1}$ to be expressed in

Table 1. White SVD coefficient matrices: block structure and eigenvalues

Matrix	Block Structure	Eigenvalues
$\widehat{Q}_{\mathrm{RX}}$	$\frac{1}{1-\sigma_i^2}\begin{pmatrix} 1 & -\sigma_i \\ -\sigma_i & 1 \end{pmatrix}$	$\{\underbrace{1/(1-\sigma_i)}_{n},\ \underbrace{1,\ldots,1}_{m},\ \underbrace{1/(1+\sigma_i)}_{n}\}$
$\widehat{Q}_{\mathrm{HACD}}$	$\frac{\sigma_i}{1-\sigma_i^2}\begin{pmatrix} \sigma_i & -1 \\ -1 & \sigma_i \end{pmatrix}$	$\{\underbrace{\sigma_i/(1-\sigma_i)}_{n},\ \underbrace{0,\ldots,0}_{m},\ \underbrace{-\sigma_i/(1+\sigma_i)}_{n}\}$
$\widehat{Q}_{\mathrm{Subpix}}$	$\frac{\sigma_i}{(1-\sigma_i^2)^2}\begin{pmatrix} -2\sigma_i & 1+\sigma_i^2 \\ 1+\sigma_i^2 & -2\sigma_i \end{pmatrix}$	$\{\underbrace{\sigma_i/(1-\sigma_i)^2}_{n},\ \underbrace{0,\ldots,0}_{m},\ \underbrace{-\sigma_i/(1+\sigma_i)^2}_{n}\}$
$\widehat{Q}_{\mathrm{CC}}$	$\frac{1}{1-\sigma_i^2}\begin{pmatrix} \sigma_i^2 & -\sigma_i \\ -\sigma_i & 1 \end{pmatrix}$	$\{\underbrace{(1+\sigma_i^2)/(1-\sigma_i^2)}_{n},\ \underbrace{\begin{matrix} 0,\ldots,0 \\ \text{or} \\ 1,\ldots,1 \end{matrix}}_{m},\ \underbrace{0,\ldots,0}_{n}\}$
$\widehat{Q}_{\mathrm{CCsym}}$	$\frac{1}{2(1-\sigma_i^2)}\begin{pmatrix} 1+\sigma_i^2 & -\sigma_i \\ -\sigma_i & 1+\sigma_i^2 \end{pmatrix}$	$\{\underbrace{\frac{1}{2}(1+\sigma_i)/(1-\sigma_i)}_{n},\ \underbrace{\frac{1}{2},\ldots,\frac{1}{2}}_{m},\ \underbrace{\frac{1}{2}(1-\sigma_i)/(1+\sigma_i)}_{n}\}$
$\widehat{Q}_{\mathrm{WTLSQ}}$		$\{\underbrace{\lambda_1^{(RX)},\ldots,\lambda_k^{(RX)}}_{k},\ \underbrace{0,\ldots,0}_{d_x+d_y-k}\}$

block diagonal form, thereby enabling the direct computation of its eigenvalues. Table 1 shows that this approach can be applied to a variety of ACD algorithms.

For the HACD algorithm, we have [8]

$$Q_{\mathrm{HACD}} = Q_{\mathrm{RX}} - \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}^{-1} \tag{17}$$

and the white SVD transformation (6) produces $\widetilde{Q}_{\mathrm{HACD}} = \widetilde{Q}_{\mathrm{RX}} - I$. Applying the Golub-Kahan permutations Π, we get $\widehat{Q}_{\mathrm{HACD}} = \widehat{Q}_{\mathrm{RX}} - I$. Thus, the eigenvectors of $\widehat{Q}_{\mathrm{RX}}$ and $\widehat{Q}_{\mathrm{HACD}}$ are identical, and the eigenvalues differ by 1.

Similarly, the coefficient matrix of the Subpixel Hyperbolic method [9]

$$Q_{\mathrm{Subpix}} = -Q_{\mathrm{RX}} \begin{pmatrix} 0 & C^T \\ C & 0 \end{pmatrix} Q_{\mathrm{RX}} \tag{18}$$

keeps its form unchanged under both white SVD and the Golub-Kahan transformations, $\widehat{Q}_{\mathrm{Subpix}} = -\widehat{Q}_{\mathrm{RX}}\, J\, \widehat{Q}_{\mathrm{RX}}$, and in these new coordinates can be viewed as the RX-transformed Jordan-Wielandt matrix J.

The Chronochrome [4] has two formulations, obtained respectively from least squares regression of D_x on D_y and D_y on D_x. These lead to [2]:

$$\widetilde{Q}_{\mathrm{CC}} = \begin{cases} \widetilde{Q}_{\mathrm{RX}} - \begin{pmatrix} I_{d_x} & 0 \\ 0 & 0 \end{pmatrix} \\ \text{or} \\ \widetilde{Q}_{\mathrm{RX}} - \begin{pmatrix} 0 & 0 \\ 0 & I_{d_y} \end{pmatrix} \end{cases}. \tag{19}$$

In case when $d_x = d_y$, both forms of $\widehat{Q}_{\mathrm{CC}}$ have permutationally equivalent block structure and identical eigenvalues, as $m = 0$. For the case $d_x \neq d_y$ matrix $\widehat{Q}_{\mathrm{CC}}$ is guaranteed to have n blocks of the form shown in Table 1, and additionally, depending on the formulation of the Chronochrome problem, an $m \times m$ block that is either I_m or 0_m, that correspond to m zero rows (columns) of the matrix $\bar{\Sigma}$ (3).

The symmetric Chronochrome is obtained by averaging the two Chronochrome matrices shown in (19). Note that this is also equivalent to $Q_{\mathrm{CCsym}} = (1/2)(Q_{\mathrm{RX}} + Q_{\mathrm{HACD}})$.

Motivated by the ordinary least squares interpretation of the Chronochrome, we recently derived an anomalous change detector using Total Least Squares [5]. We showed that Whitened Total Least Squares (WTLSQ) is equivalent to Optimized Covariance Equalization [6], as well as to the Canonical Correlation Analysis-based Multivariate Alteration Detection [7]. The whitened coefficient matrix $\widetilde{Q}_{\mathrm{WTLSQ}}$ can be expressed

$$\widetilde{Q}_{\mathrm{WTLSQ}} \equiv \widetilde{B}_k (\widetilde{B}_k^T\, \widetilde{Q}_{\mathrm{RX}}^{-1}\, \widetilde{B}_k)^{-1}\, \widetilde{B}_k^T, \tag{20}$$

where B_k has the effect of retaining the k largest eigenvalues of $\widetilde{Q}_{\mathrm{RX}}$ and setting the remainder of the eigenvalues to zero. Although it is possible to apply a Golub-Kahan permutation to $\widetilde{Q}_{\mathrm{WTLSQ}}$, that is not necessary since we already have the eigenvalue spectrum from Q_{RX}.

In Table 1 we present the block structure and the analytical expressions of the eigenvalues of the discussed matrices as the functions of the singular values $\sigma_i,\ i = 1, 2, \ldots, n$ of the whitened covariance matrix. Since $0 \leq \sigma_i \leq 1$, it is clear that the eigenvalues for RX, WTLSQ, and all of the Chronochrome detectors are non-negative; whereas the eigenvalues for HACD and Subpixel HACD take both positive and negative values.

3. CONCLUSIONS

We presented a methodology that provides the eigenvalue spectrum for a wide range of quadratic anomalous change detectors. Table 1 summarizes these results, and Fig. 1 illustrates them. Although their eigen*values* differ, we find that RX, HACD, Subpixel HACD, symmetrized Chronochrome, and WTLSQ share the same eigen*vectors*. The eigen*vectors* for the two variants of Chronochrome defined in (19) are different, and are different from each other, even though they share many (but not all, unless $d_x = d_y$) eigenvalues. We demonstrated that it is sufficient to compute SVD of the whitened cross covariance matrix of the data in order to almost immediately obtain highly structured sparse matrices (and their eigenvalue spectra) of the coefficient matrices of these ACD algorithms in the white SVD-transformed coordinates. Converting to the original non-white coordinates, these eigenvalues will be modified in magnitude but not in sign.

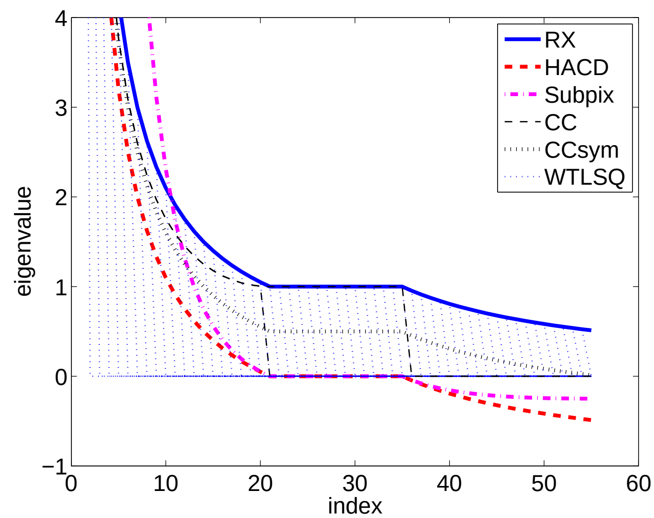

Fig. 1. Eigenvalue spectrum for various ACD algorithms, with eigenvalues arranged in decreasing order. Here $d_x = 20$ and $d_y = 35$, so $n = 20$ and $m = 15$. We have taken $\sigma_i = (n+1-i)/(n+1)$ so that $0 < \sigma_i < 1$ uniformly for $i = 1, 2, \ldots, n$. There are two Chronochrome (CC) curves; they agree on the first n indices and on the last n indices, but differ for the middle m. Only HACD and Subpixel HACD exhibit negative eigenvalues. The WTLSQ detector agrees with RX for the first k eigenvalues, and is zero thereafter.

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