Exploring New Leptonic Physics With Effective Operators

A model independent survey of lepton number violation and related topics

James Jenkins

Department of Physics and Astronomy
Northwestern University

INFO 07–Santa Fe

Outline

1. Introduction
   - Why Physics Beyond the Standard Model?
   - The Effective Operator Approach
   - Lepton Number Violation (LNV) and Majorana particles
2. LNV Operators and the Scale of New Physics
   - Motivation
   - An extended example
   - Λ The scale of new physics
3. Constraints and predictions
   - Oscillation phenomenology
   - Neutrinoless Double β-decay
   - Other Rare Decays
   - Collider Signatures

Looking beyond the Standard Model

Despite the success of the Standard Model we expect new high scale physics

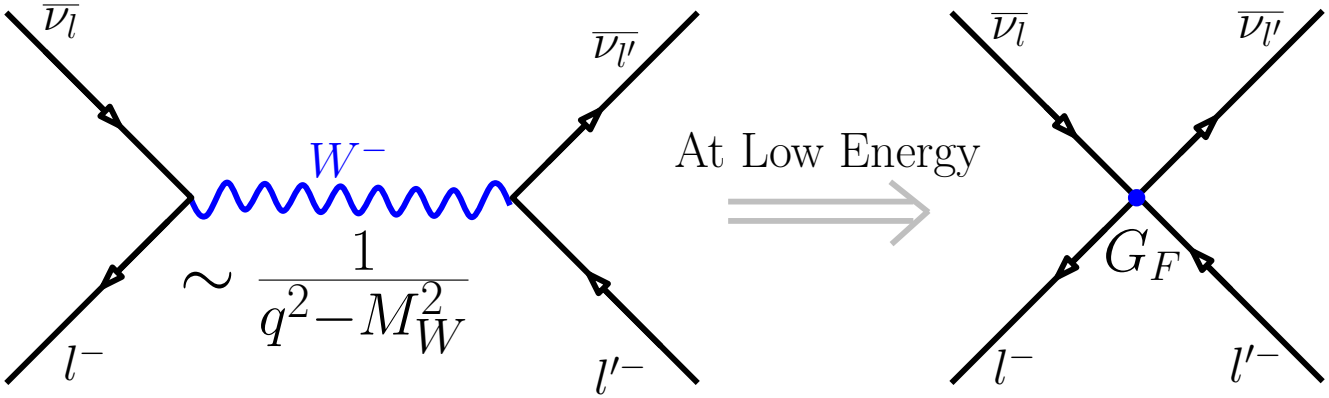
Why new physics?

- Direct evidence
  - Neutrino masses
  - Dark matter/Dark energy
  - Baryon-Antibaryon asymmetry of the universe
- Indirect evidence
  - Stabilize Higgs mass
  - Particle mass spectrum (neutrinos)
  - Unification of gauge couplings

Many models exist... How to distinguish them?

Effective Operators

What does new physics look like to a low energy observer?



Effective Lagrangian at low energy

$$\mathcal{L}_{eff} = \mathcal{L}_{SM}(d = 4) + \sum_n \frac{1}{\Lambda^n} \mathcal{O}^{4+n}$$

Majorana particles

Neutrinos are neutral and can therefore be their own antiparticle.

Particle/antiparticle distinctions are meaningless for Majorana fields

The Majorana Condition

$$\psi = \eta \psi^c$$

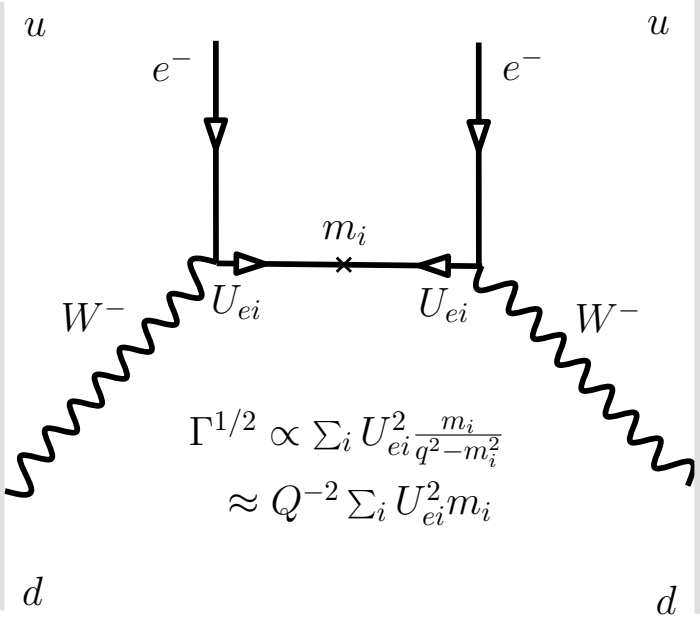
Mass Terms

$$\begin{aligned} \Delta\mathcal{L}_D &= m_D(\overline{e}_R e_L + \overline{e}_L e_R) \\ \Delta\mathcal{L}_M &= \frac{1}{2} m_M(\overline{\nu^c}\nu + \overline{\nu}\nu^c) \end{aligned}$$

Example

$$\begin{aligned} \langle e_R \mid \Delta\mathcal{L}_D \mid e_L \rangle &= m_D \\ \langle \nu^c \mid \Delta\mathcal{L}_M \mid \nu \rangle &= m_M \end{aligned}$$

Neutrinoless double beta decay



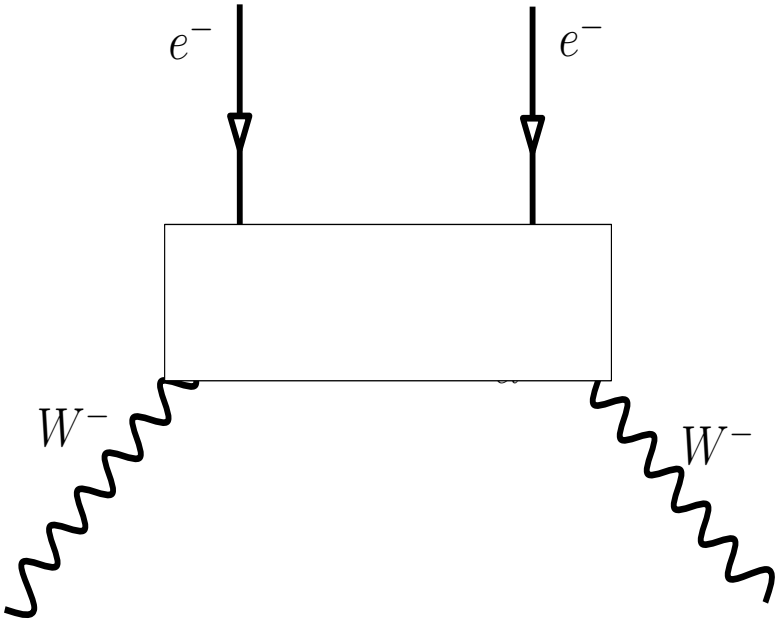
- A $\Delta L = 2$ process that proceeds via
  - Majorana neutrino exchange – $\beta\beta0\nu$ is only realistic probe of LNV
  - Some (as yet) unknown high scale process
- Experiments typically set upper limits on m_{ee}

Assuming the dimension 9 operator ($eeuu\overline{dd}$) we can extract the mass scale Λ probed by $\beta\beta0\nu$ (assume $m_{ee} = 0.05$ eV)

$$\frac{1}{\Lambda^5} \approx \frac{G_f^2 m_{ee}}{Q^2} \quad \Rightarrow \Lambda \approx 3.3 \text{ TeV}$$

Majorana neutrinos and Lepton Number Violation

The Black Box and extended Black Box theorems



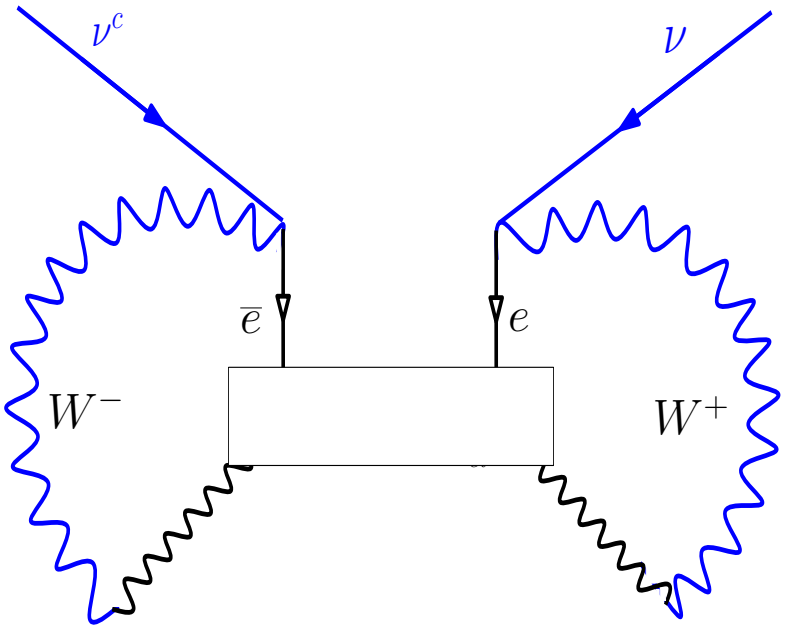
- Black Box Theorem Schechter & Valle, 1982
  - LNV operators yield Majorana neutrino mass
  - Majorana neutrino mass yields effective LNV operators
- $m_{ee}^{eff} = \sum_i (\mathcal{O}_i + m_{ee}^{\mathcal{O}_i})$
  Exact cancelations are possible but require fine tuning

Extended Black Box Theorem: Assuming 3 ν mixing & arbitrary LNV, $m_{ee} = 0$ is *not* consistent with oscillation data!

Hirsch, Kovalenko & Schmidt 2006

Majorana neutrinos and Lepton Number Violation

The Black Box and extended Black Box theorems



- Black Box Theorem Schechter & Valle, 1982
  - LNV operators yield Majorana neutrino mass
  - Majorana neutrino mass yields effective LNV operators
- $m_{ee}^{eff} = \sum_i (\mathcal{O}_i + m_{ee}^{\mathcal{O}_i})$
  Exact cancelations are possible but require fine tuning

Extended Black Box Theorem: Assuming 3 ν mixing & arbitrary LNV, $m_{ee} = 0$ is *not* consistent with oscillation data!

Hirsch, Kovalenko & Schmidt 2006

Motivation

I perform an exhaustive survey of *all* LNV nonrenormalizable operators up to dimension 11

Questions to answer

- Is $\beta\beta0\nu$ always the best probe of LNV? No
  - Should we look for other signals of LNV? Where?
  - What is the nature of models predicting large LNV?
- If the neutrino Majorana mass is highly suppressed can we still expect to observe $\beta\beta0\nu$ in next generation experiments? Yes
  - What is the nature of these models?
  - What are the other observable features of these models?
- What can we generally learn about LNV models from future experiments/observations? much...

Plan of attack

1. *Categorize* all effective LNV operators up to dimension 11
   - All $SU(3)_c \times SU(2)_L \times U(1)_y$ invariant SM operators

     Babu & Leung, 2001
   - All possible Lorentz contractions
2. *Calculate* radiatively generated neutrino mass
3. *Extract* the scale (Λ) of new physics from observed neutrino mass $(0.05 - 1)\ \mathrm{eV}$
4. *Check* for additional constraints
5. *Predict* possible modes of future observation
6. *Select interesting* operators for further detailed study
   - Highly suppressed m_ν and $\beta\beta0\nu \Rightarrow$ Tree level phenomenology
   - Suppressed m_ν, but enhanced tree level $\beta\beta0\nu$

Plan of attack

1. *Categorize* all effective LNV operators up to dimension 11
   - All $SU(3)_c \times SU(2)_L \times U(1)_y$ invariant SM operators

     Babu & Leung, 2001
   - All possible Lorentz contractions
2. *Calculate* radiatively generated neutrino mass
3. *Extract* the scale (Λ) of new physics from observed neutrino mass $(0.05 - 1)\ \mathrm{eV}$
4. *Check* for additional constraints
5. *Predict* possible modes of future observation
6. *Select interesting* operators for further detailed study
   - Highly suppressed m_ν and $\beta\beta0\nu \Rightarrow$ Tree level phenomenology
   - Suppressed m_ν, but enhanced tree level $\beta\beta0\nu$

Plan of attack

1. *Categorize* all effective LNV operators up to dimension 11
   - All $SU(3)_c \times SU(2)_L \times U(1)_y$ invariant SM operators

     Babu & Leung, 2001
   - All possible Lorentz contractions
2. *Calculate* radiatively generated neutrino mass
3. *Extract* the scale (Λ) of new physics from observed neutrino mass $(0.05 - 1)\ \mathrm{eV}$
4. *Check* for additional constraints
5. *Predict* possible modes of future observation
6. *Select interesting* operators for further detailed study
   - Highly suppressed m_ν and $\beta\beta0\nu \Rightarrow$ Tree level phenomenology
   - Suppressed m_ν, but enhanced tree level $\beta\beta0\nu$

Plan of attack

1. *Categorize* all effective LNV operators up to dimension 11
   - All $SU(3)_c \times SU(2)_L \times U(1)_y$ invariant SM operators

     Babu & Leung, 2001
   - All possible Lorentz contractions
2. *Calculate* radiatively generated neutrino mass
3. *Extract* the scale (Λ) of new physics from observed neutrino mass $(0.05 - 1)\ \mathrm{eV}$
4. *Check* for additional constraints
5. *Predict* possible modes of future observation
6. *Select interesting* operators for further detailed study
   - Highly suppressed m_ν and $\beta\beta0\nu \Rightarrow$ Tree level phenomenology
   - Suppressed m_ν, but enhanced tree level $\beta\beta0\nu$

Plan of attack

1. *Categorize* all effective LNV operators up to dimension 11
   - All $SU(3)_c \times SU(2)_L \times U(1)_y$ invariant SM operators

     Babu & Leung, 2001
   - All possible Lorentz contractions
2. *Calculate* radiatively generated neutrino mass
3. *Extract* the scale (Λ) of new physics from observed neutrino mass $(0.05 - 1)\ \mathrm{eV}$
4. *Check* for additional constraints
5. *Predict* possible modes of future observation
6. *Select interesting* operators for further detailed study
   - Highly suppressed m_ν and $\beta\beta0\nu \Rightarrow$ Tree level phenomenology
   - Suppressed m_ν, but enhanced tree level $\beta\beta0\nu$

Plan of attack

1. *Categorize* all effective LNV operators up to dimension 11
   - All $SU(3)_c \times SU(2)_L \times U(1)_y$ invariant SM operators

     Babu & Leung, 2001
   - All possible Lorentz contractions
2. *Calculate* radiatively generated neutrino mass
3. *Extract* the scale (Λ) of new physics from observed neutrino mass $(0.05 - 1)\ \mathrm{eV}$
4. *Check* for additional constraints
5. *Predict* possible modes of future observation
6. *Select interesting* operators for further detailed study
   - Highly suppressed m_ν and $\beta\beta0\nu \Rightarrow$ Tree level phenomenology
   - Suppressed m_ν, but enhanced tree level $\beta\beta0\nu$

Ex. $\mathcal{O}_2 = L^i L^j L^k e_R^c H^l \epsilon_{ij} \epsilon_{kl}$

A simple four fermion dimension 7 operator

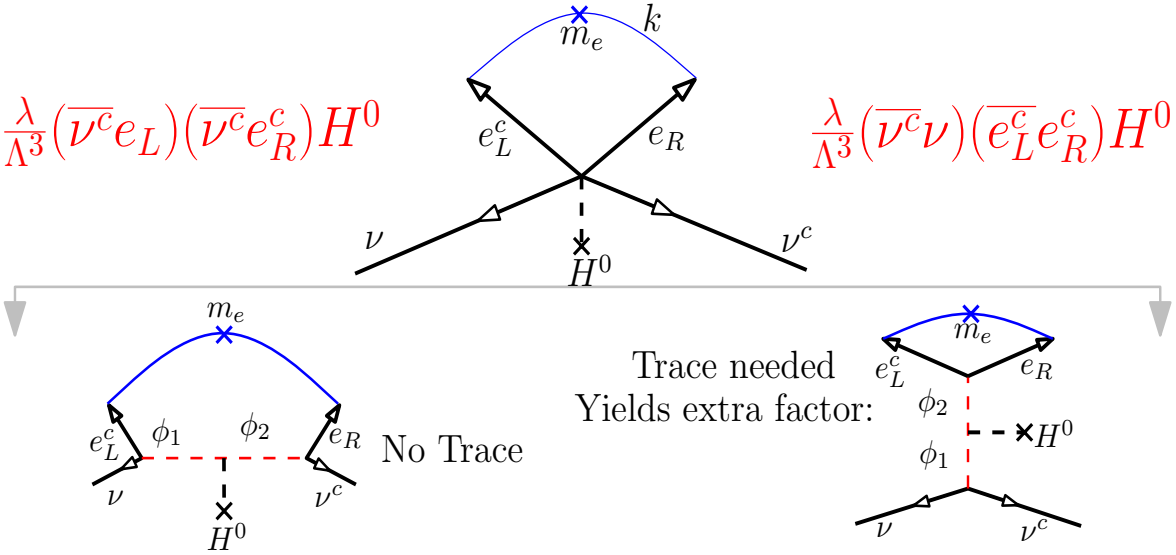
Characteristics

- $\Delta L = 2$ (lepton number violating)
- Suppressed by $1/\Lambda^3$
- Carrying out the $SU(2)_L$ contractions:
  $\nu_a (e_L)_b \nu_c (e_R^c)_d H^0 - (e_L)_a \nu_b \nu_c (e_R^c)_d H^0 - \nu_a (e_L)_b (e_L)_c (e_R^c)_d H^+ + (e_L)_a \nu_b (e_L)_c (e_R^c)_d H^+$
- Possible Lorentz contractions of fermions
  - $(\overline{L^c}^i L^j)(\overline{L^c}^k e^c)$, $(\overline{L^c}^i L^k)(\overline{L^c}^j e^c)$, $(\overline{L^c}^i e^c)(\overline{L^c}^j L^k)$
  - $(\overline{L^c}^i \sigma^{\mu\nu} L^j)(\overline{L^c}^k \sigma_{\mu\nu} e^c)$, $(\overline{L^c}^i \sigma^{\mu\nu} L^k)(\overline{L^c}^j \sigma_{\mu\nu} e^c)$, $(\overline{L^c}^i \sigma^{\mu\nu} e^c)(\overline{L^c}^j \sigma_{\mu\nu} L^k)$

Ex. $\mathcal{O}_2$: Induced neutrino mass

$\frac{\lambda}{\Lambda^3}(\overline{\nu^c} e_L)(\overline{\nu^c} e_R^c) H^0$

$\frac{\lambda}{\Lambda^3}(\overline{\nu^c} \nu)(\overline{e_L^c} e_R^c) H^0$



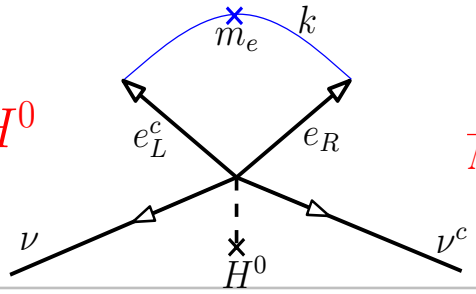
- Easy to determine Lorentz structure from diagram
- Only 4 distinct mass diagrams
- Easy to expand vertices
- Replace H^0 with VEV

$$\delta m_\nu = \bar{u}_p \frac{\lambda v}{\Lambda^3} \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{\not{k}}{k^2} m_e \frac{\not{k}}{k^2} u_p \times \mathcal{O}(1) \text{ factor}$$

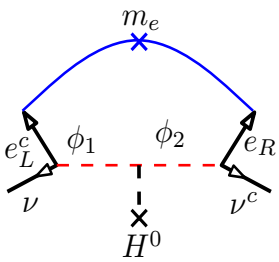
$$= \frac{\lambda m_e}{16\pi^2} \frac{v}{\Lambda} \times \mathcal{O}(1) \text{ factor}$$

Ex. $\mathcal{O}_2$: Induced neutrino mass

$$\frac{\lambda}{\Lambda^3}(\overline{\nu^c} e_L)(\overline{\nu^c} e_R^c) H^0$$

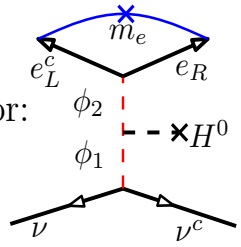


$$\frac{\lambda}{\Lambda^3}(\overline{\nu^c} \nu)(\overline{e_L^c} e_R^c) H^0$$



No Trace

Trace needed
Yields extra factor:



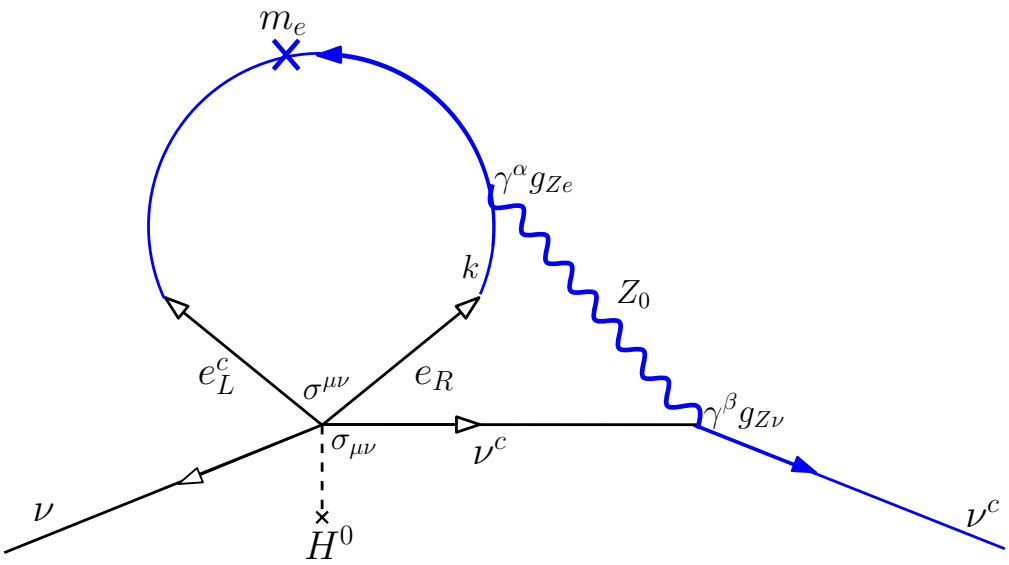
- Scalar-Scalar No Tr: 1
- Scalar-Scalar Tr: 4
- Tensor-Tensor No Tr: 12
- Tensor-Tensor Tr: **0**

$$Tr[\sigma_{\mu\nu}] = 0$$

Assuming a coupling constant $\lambda = 1$ we extract:

$$\Lambda = 5.5 \times 10^7 \left(\frac{0.05\ \mathrm{eV}}{m_\nu}\right)\left(\frac{m_e}{m_\tau}\right)\ \mathrm{TeV}$$

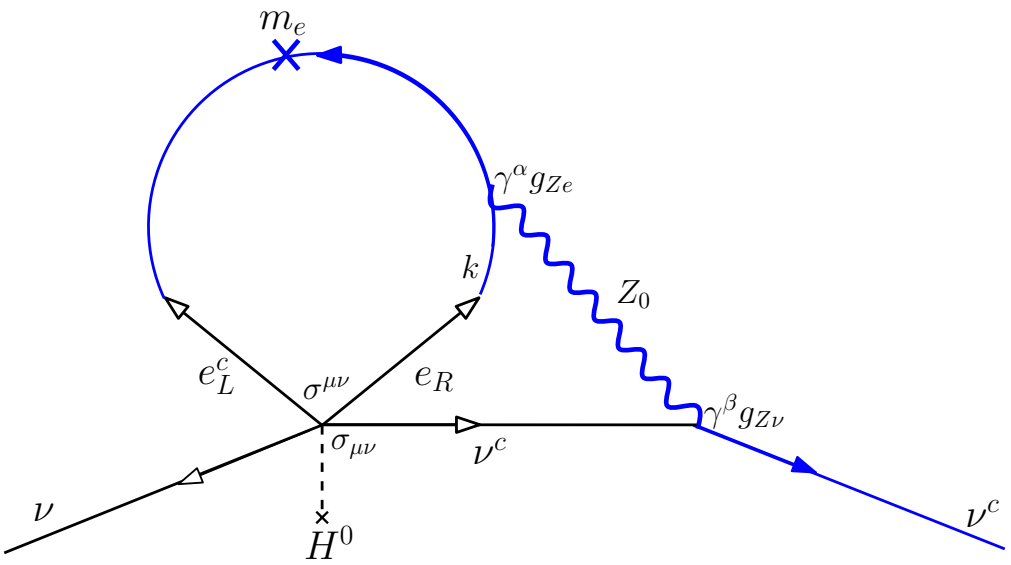
Ex. $\mathcal{O}_2$: 2 loop evaluation of the vanishing operator



$$\bar{u}(p)\int \frac{d^4q d^4k}{(2\pi)^8} g_{\nu Z}\gamma^\alpha \frac{\eta_{\alpha\beta}}{q^2}\frac{(\not{p}-\not{q})}{(p-q)^2}\sigma_{\mu\nu}\frac{\lambda}{\Lambda^3}\frac{Tr[\sigma^{\mu\nu}(\not{k}-\not{q})\gamma^\beta g_{eZ}\not{k} m_e \not{k}]}{(k-q)^2k^2k^2}u(p)$$

$$= \frac{48\lambda m_e v}{(16\pi^2)^2\Lambda} g_{\nu Z} g_{eZ} ln\left(\frac{\Lambda}{v}\right)$$

Ex. $\mathcal{O}_2$: 2 loop evaluation of the vanishing operator



Assuming a coupling constant $\lambda = 1$ we extract:

$$\Lambda = 1.5 \times 10^6 \text{ TeV}$$

This is too hard!!!

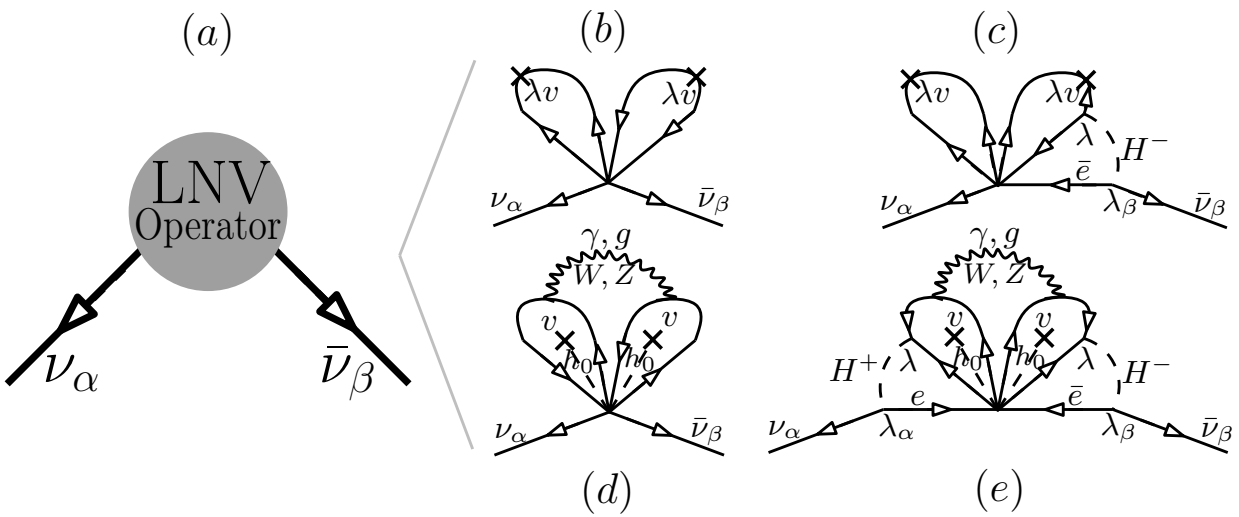
Doing this for all 129 operators is not necessary

Estimate amplitudes based of the assumptions/rules:

- Include trivial numerical factors: g_i, λ_{ij}, Λ^{-n}, v and color enhancements. Assume operator couplings $\lambda \approx 1$.
- Assume SM fields massless at 0th order... Add as needed.
- Obtain loop divergences from power counting.
- Include phase space suppressions $16\pi^2$ for each loop
  adding loops to enhance rates for $\Lambda < 3\ \text{TeV}$ is not efficient.
- Assume tree level intermediate virtual states at characteristic momentum transfer Q.
  Gauge boson $\sim (Q^2 - M_g^2)^{-1}$, Fermion $\sim (Q - M_f)^{-1}$
- Neglect Lorentz structure and logarithmic divergences.

Miscellaneous diagrams

These neutrino self energies illustrate many general features



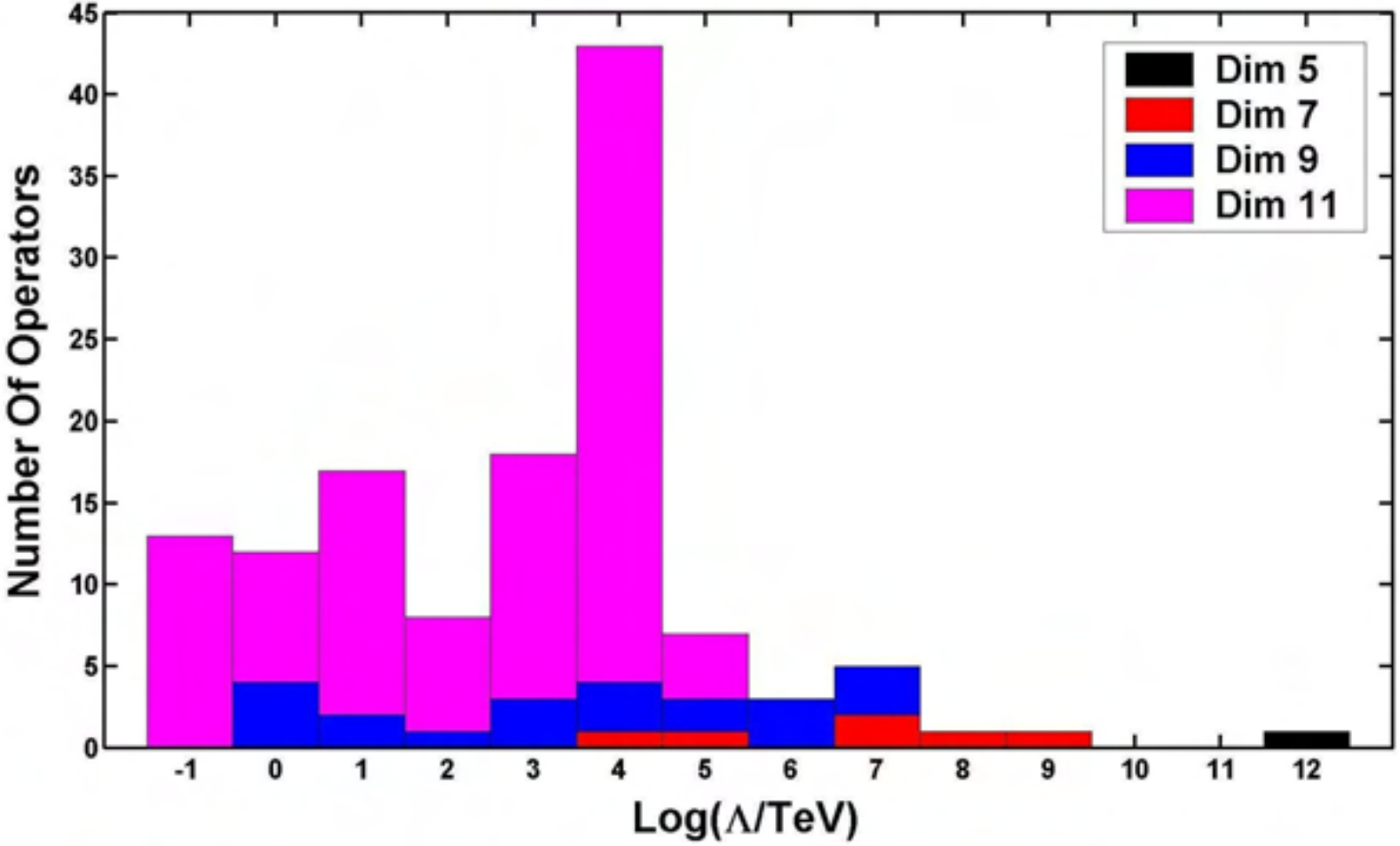
- Different dim. and loop #
- (b) $\propto \lambda_i^{(3)} \lambda_j^{(3)}$
- (d) λ independent!!!
- (c) $\propto \lambda_\alpha$
- (e) $\propto \lambda_\alpha \lambda_\beta$

Despite the dimension of the underlying operators canceling divergences yields:

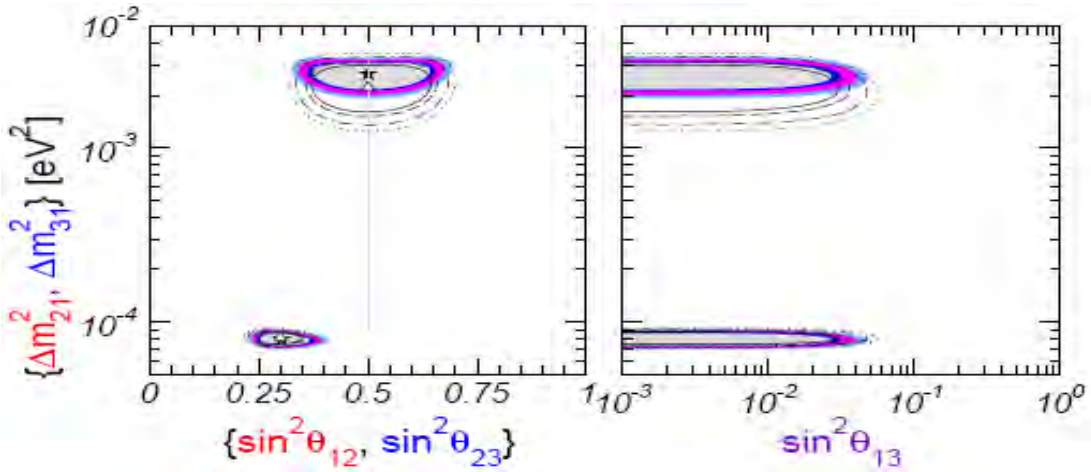
$$m_{\alpha\beta} \sim \xi_{\alpha\beta} \frac{1}{\Lambda} L_\alpha L_\beta H H$$

Dim 5!

Λ Scale distribution



Implied neutrino mass textures



$$M \propto \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

20 of these

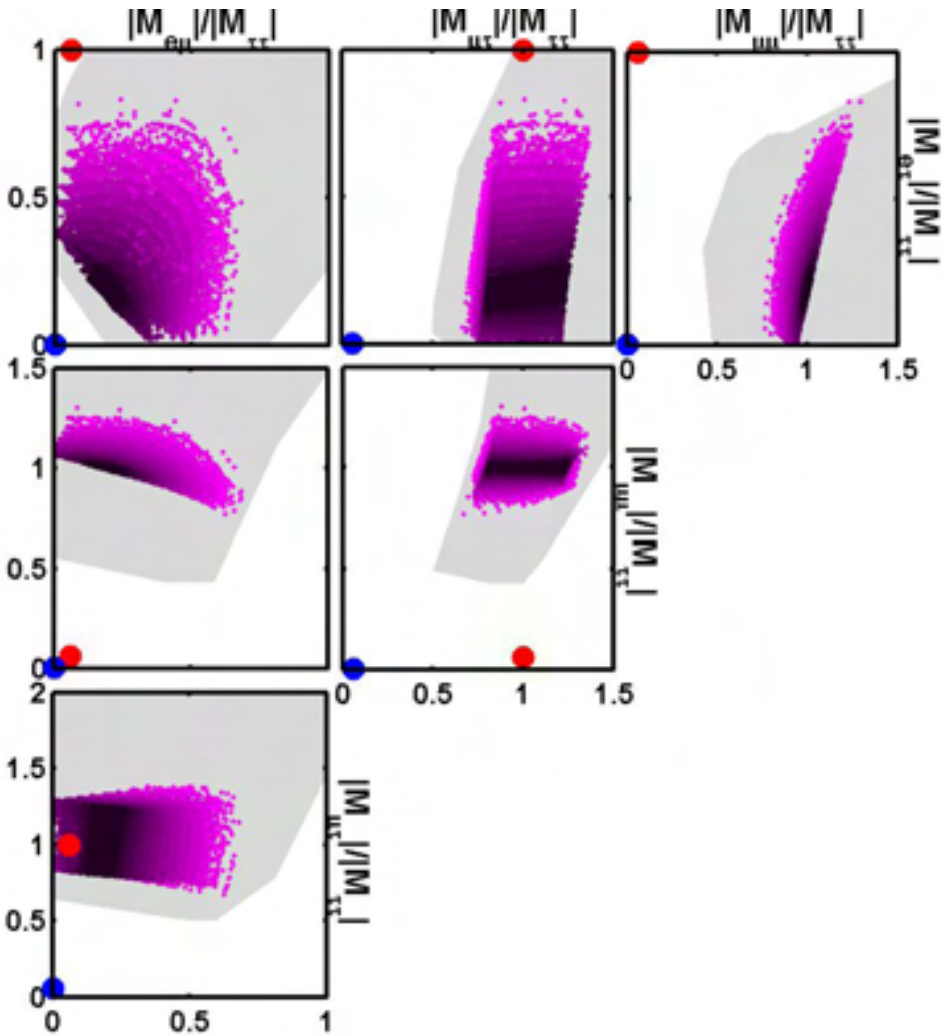
$$M \propto \begin{pmatrix} \lambda_e & \lambda_\mu & \lambda_\tau \\ \lambda_\mu & \lambda_\mu & \lambda_\tau \\ \lambda_\tau & \lambda_\tau & \lambda_\tau \end{pmatrix}$$

3 of these

$$M \propto \begin{pmatrix} \lambda_e\lambda_e & \lambda_e\lambda_\mu & \lambda_e\lambda_\tau \\ \lambda_e\lambda_\mu & \lambda_\mu\lambda_\mu & \lambda_\mu\lambda_\tau \\ \lambda_e\lambda_\tau & \lambda_\mu\lambda_\tau & \lambda_\tau\lambda_\tau \end{pmatrix}$$

- 9 parameters in a Majorana mixing matrix
  $\{m_1, m_2, m_3, \theta_{12}, \theta_{23}, \theta_{13}, \delta, \phi_2, \phi_3\}$
- Oscillation Constraints
  - Solar: $\Delta^2_S = m^2_2 - m^2_1$, $\theta_S = \theta_{12}$
  - Atmospheric: $|\Delta^2_A| = |m^2_3 - m^2_2|$, $\theta_A = \theta_{23}$
  - Reactor: $\theta_R = \theta_{13}$
- $m_{ee} \approx 0$ implies Normal hierarchy

Implied neutrino mass textures (numerical results)



- Require $m_{ee} < 10^{-4}$ eV
- all phases vary freely
- 95% confidence limits
- θ_{13} varies from $0^\circ - 14^\circ$

$$M \propto \begin{pmatrix} \lambda_e & \lambda_\mu & \lambda_\tau \\ \lambda_\mu & \lambda_\mu & \lambda_\tau \\ \lambda_\tau & \lambda_\tau & \lambda_\tau \end{pmatrix}$$

$$M \propto \begin{pmatrix} \lambda_e\lambda_e & \lambda_e\lambda_\mu & \lambda_e\lambda_\tau \\ \lambda_e\lambda_\mu & \lambda_\mu\lambda_\mu & \lambda_\mu\lambda_\tau \\ \lambda_e\lambda_\tau & \lambda_\mu\lambda_\tau & \lambda_\tau\lambda_\tau \end{pmatrix}$$

Golden Channel

$$\ell_\alpha \ell_\beta \to d_\mu d_\nu \bar{u}_\sigma \bar{u}_\rho$$

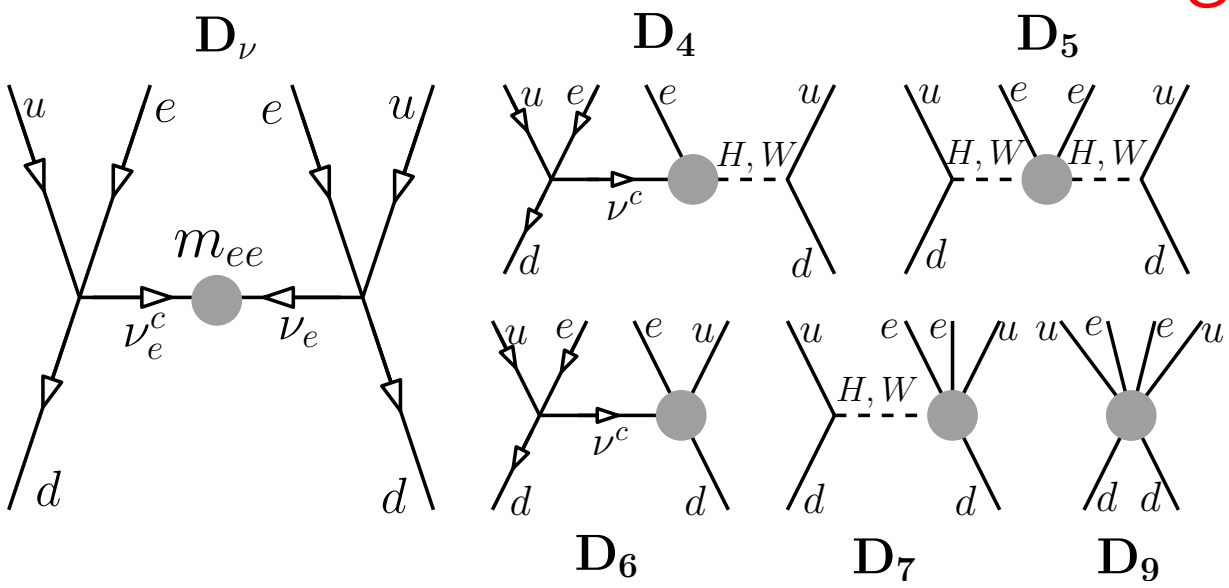
Other possibilities

- External state neutrinos: $\ell_\alpha \leftrightarrow \nu_\alpha$
  Potential for β-beam facilities?
- More external state leptons $\Rightarrow$ neutrino always included
- Only two external state quarks: $\ell_\alpha \ell_\beta \to d_\mu \bar{u}_\sigma + W^-$
- Rare W and Z boson decays

Each mode is experimentally challenging in its own way... But the golden channel is the most promising!

Neutrinoless double β-decay

Contributing diagrams and assumptions



General Observations

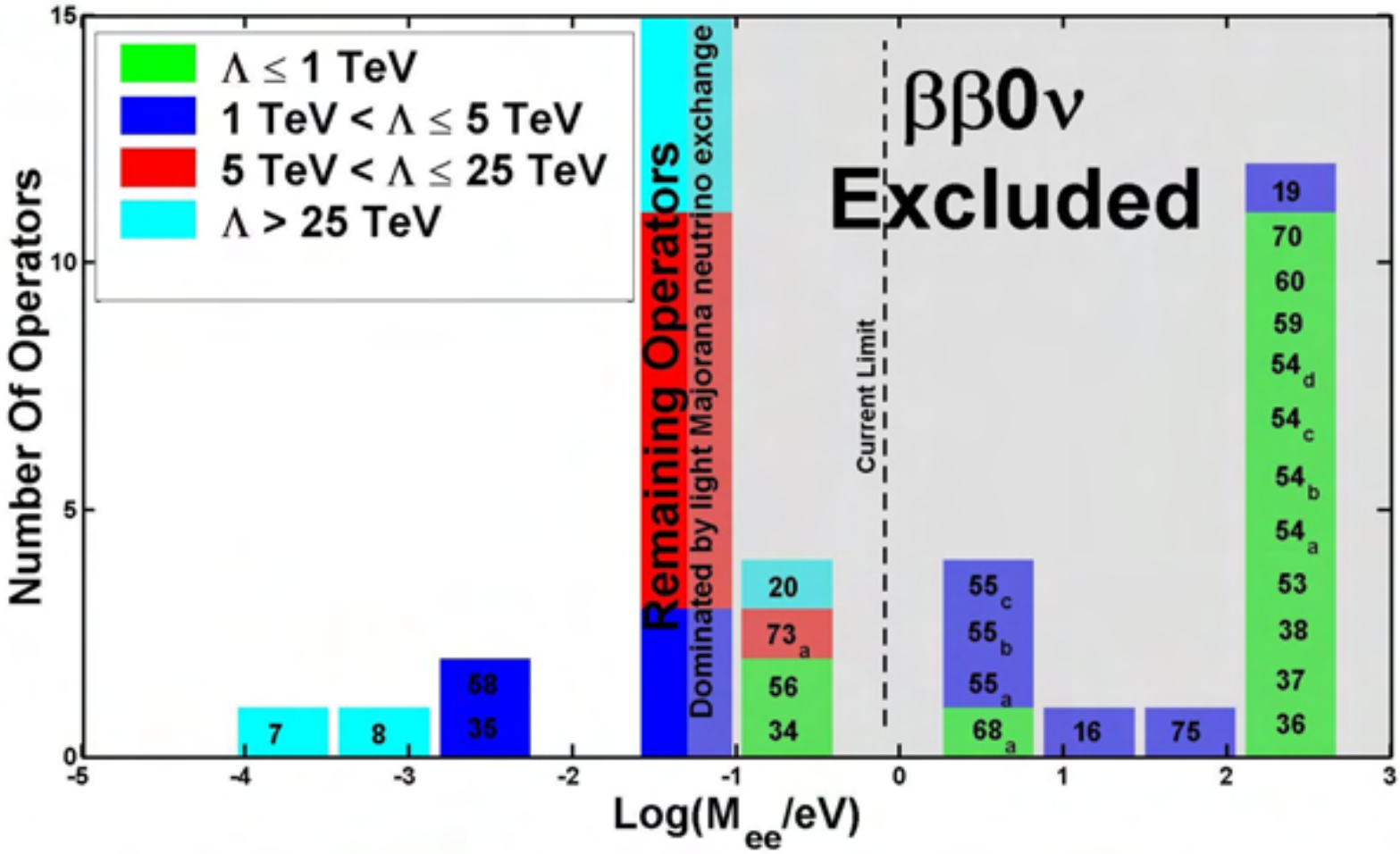
- Tree level contributions are dominant at sufficiently low scales!
  - D_9 dominates low Λ
  - D_ν dominates high Λ
  - D_6 dominates middle Λ
    - ν yields Q^{-1} factor
- $D_\nu > D_4$ in most cases

- Neglect variations in nuclear matrix element
- Diagrams are added incoherently due to chiral structure

$$m_{ee}^{eff} = \frac{Q^2}{G_f^2 |V_{ud}|^2} \sqrt{\sum_i \mathcal{A}_{D_i}^2}$$

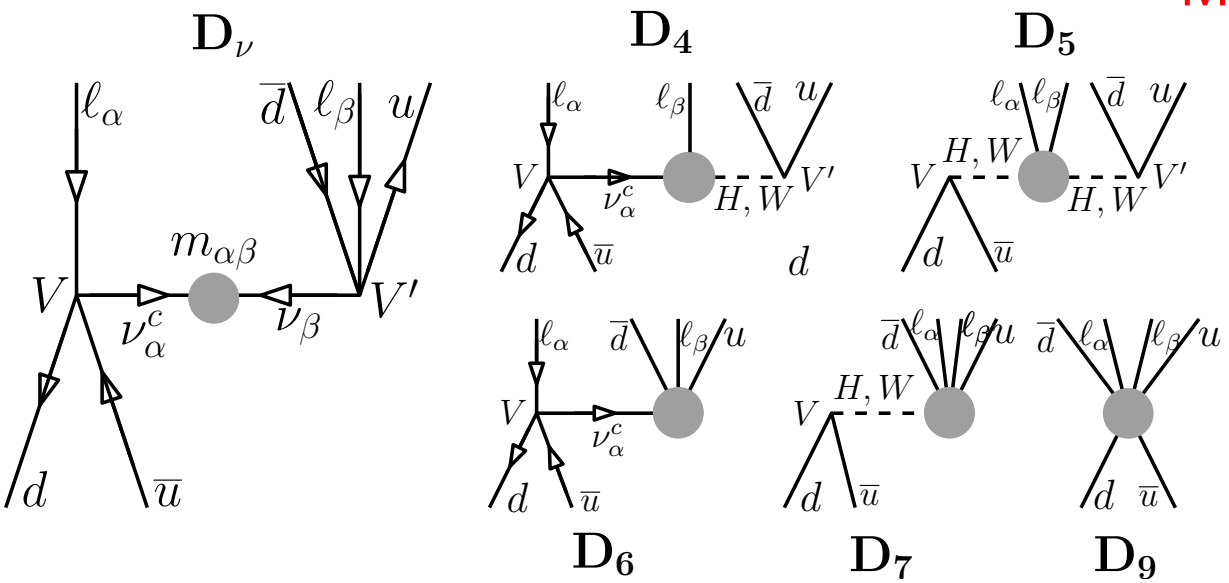
Neutrinoless double β-decay

m_{ee} distribution



Meson and τ decays

Contributing diagrams and assumptions



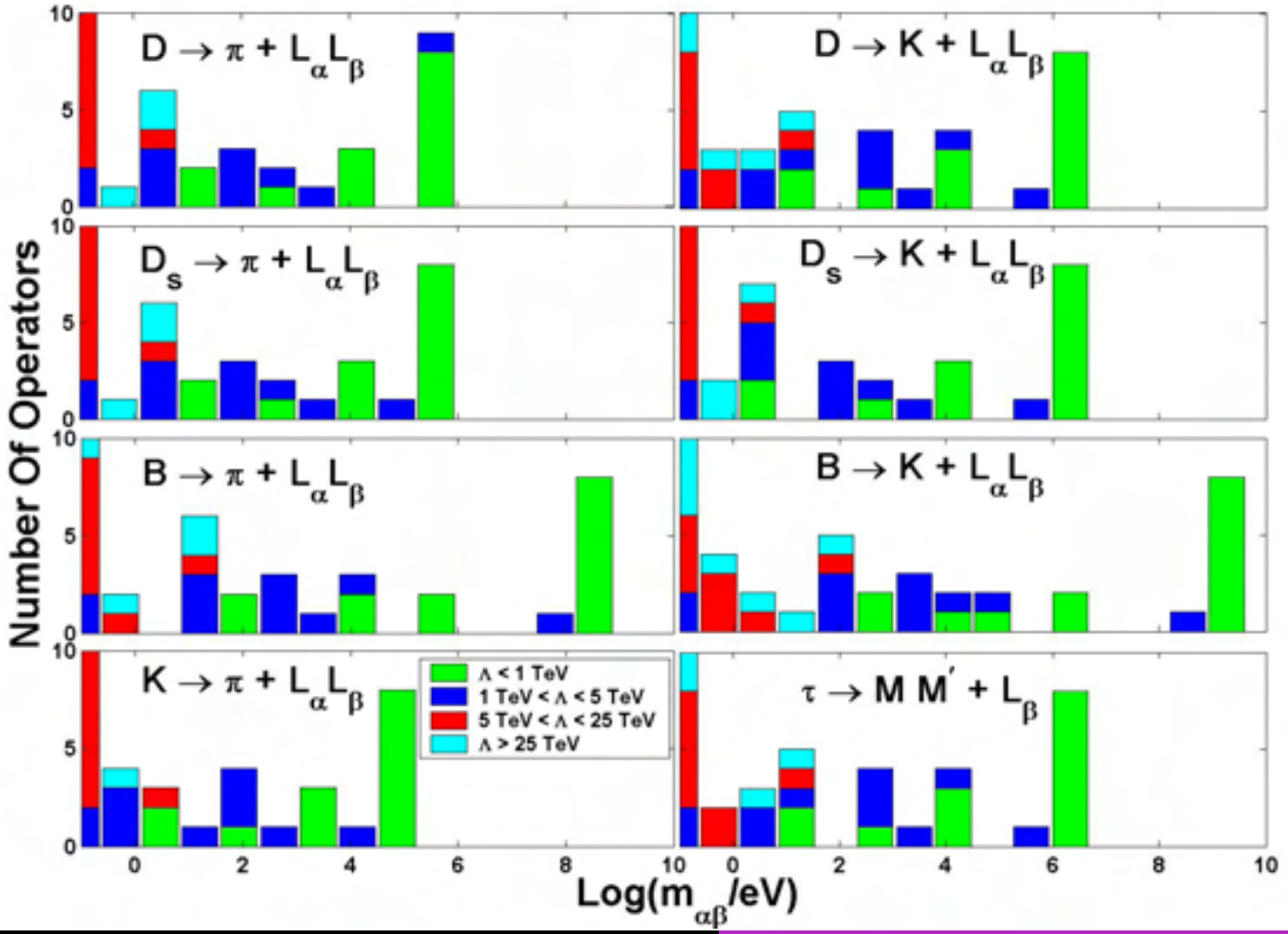
Modification

- Use crossing symmetry relate amplitudes
- CKM matrix element V
- Include new Q
- Modify phase space
- Lorentz structure: $\sigma_{\mu\nu}$

- Neglect variations in hadronic matrix element
- Diagrams are added incoherently due to chiral structure
- $m_{\alpha\beta}$ bounds calculated for dominant decays

Meson and τ decays

$m_{\alpha\beta}$ distributions



International Linear Collider (ILC)

Consider ILC in e^-e^--mode at 1 TeV COM energy and 100 fb^{-1} characteristic integrated luminosity.

$$e^-e^- \to 4j + (\text{no missing energy})$$

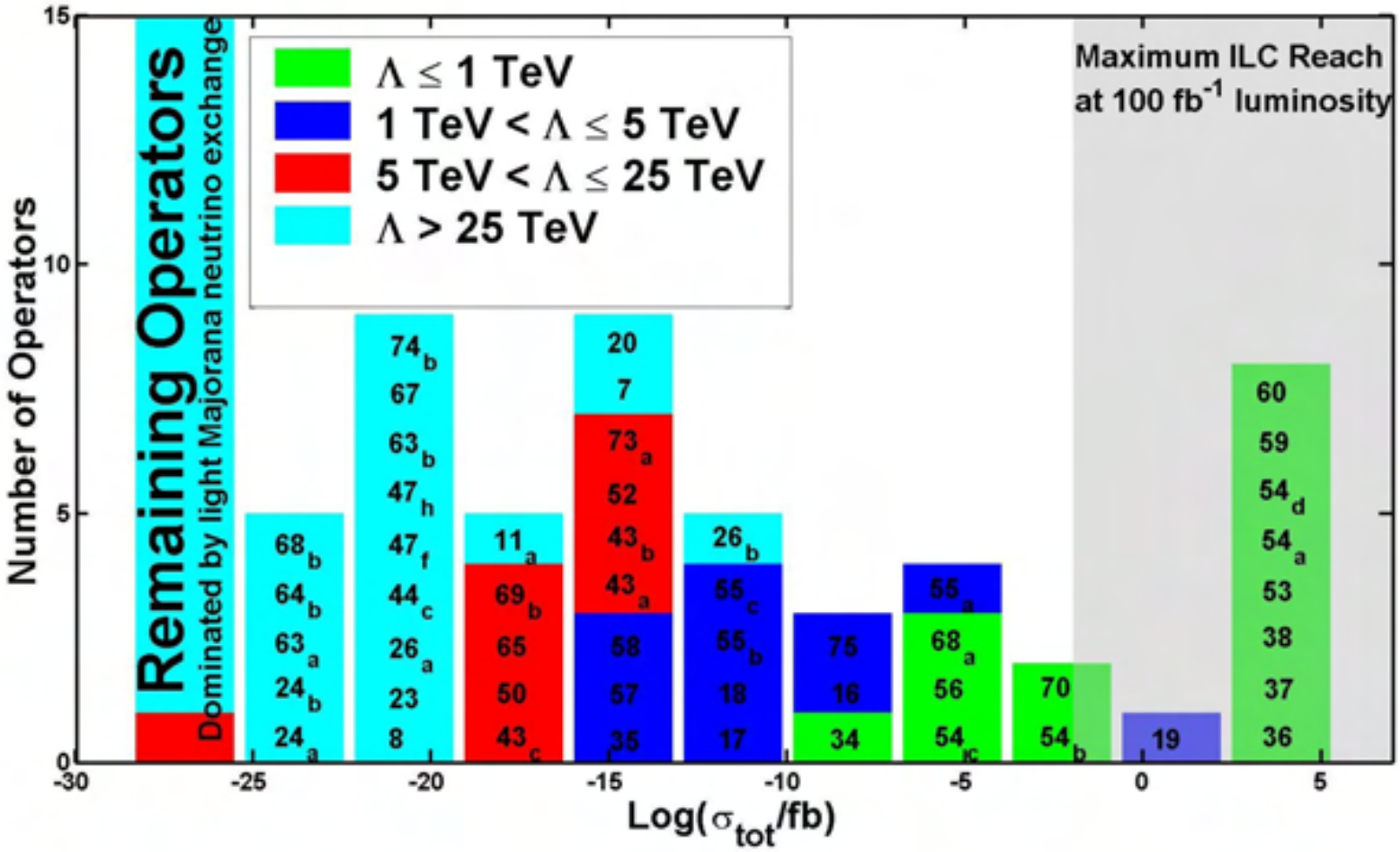
Characteristics and assumptions

- Resonance production is important... Assume $\Lambda > \mathrm{TeV}$
- Structurally equivalent to $0\nu\beta\beta$ process
- Little or no SM background
- $\gamma\gamma$ collider possibilities
- $\mu^-\mu^-$ collider possibilities

Final state leptons also possible: $e^-e^- \to \mu\nu + 2j$

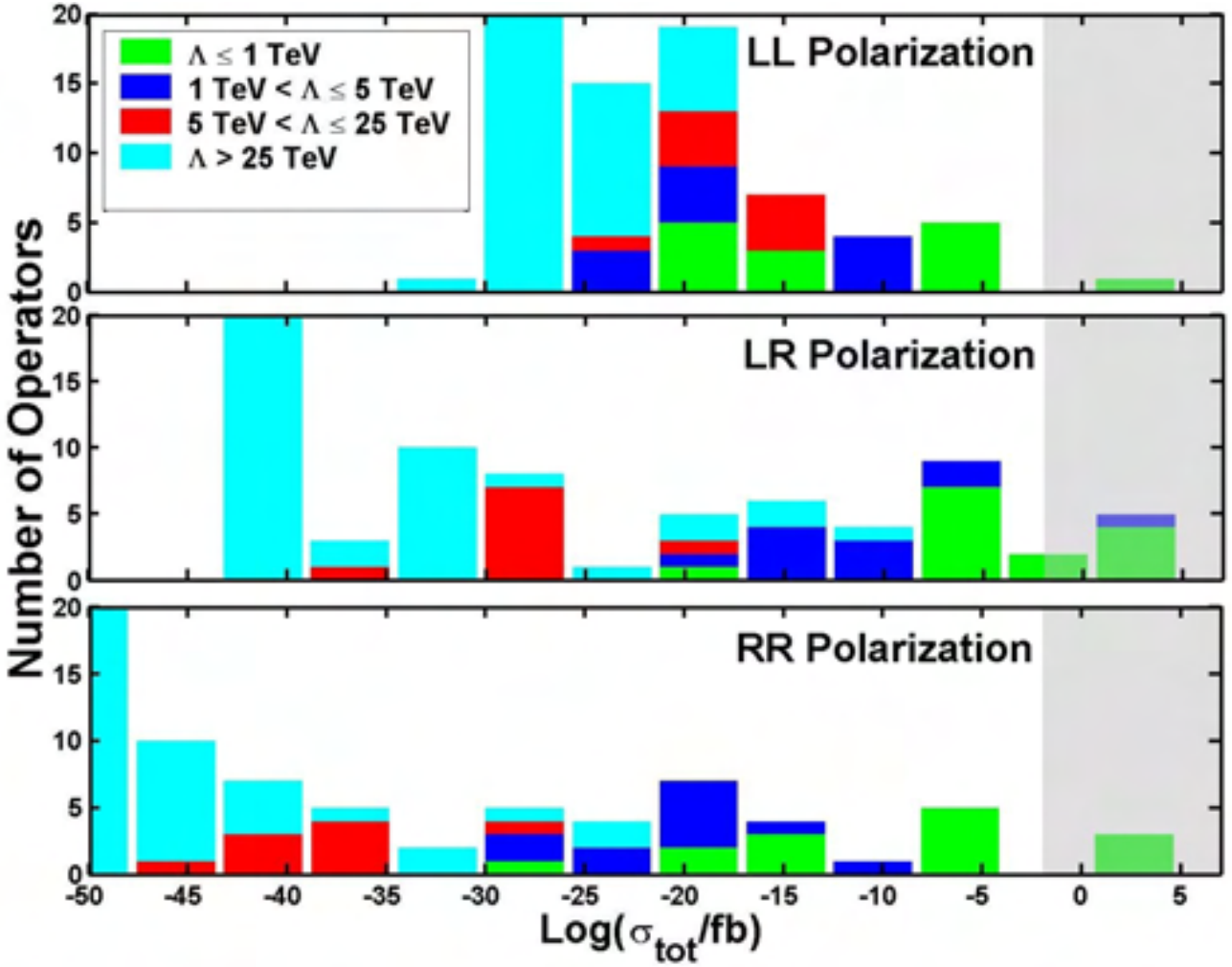
ILC in e^-e^- mode

Cross-section distribution

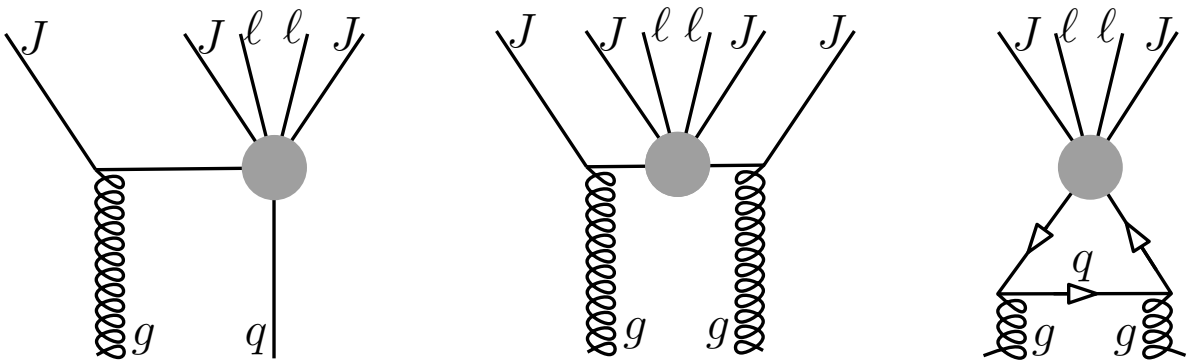


ILC in e^-e^- mode

Cross-section distribution assuming polarized initial state



Large Hadron Collider (LHC)



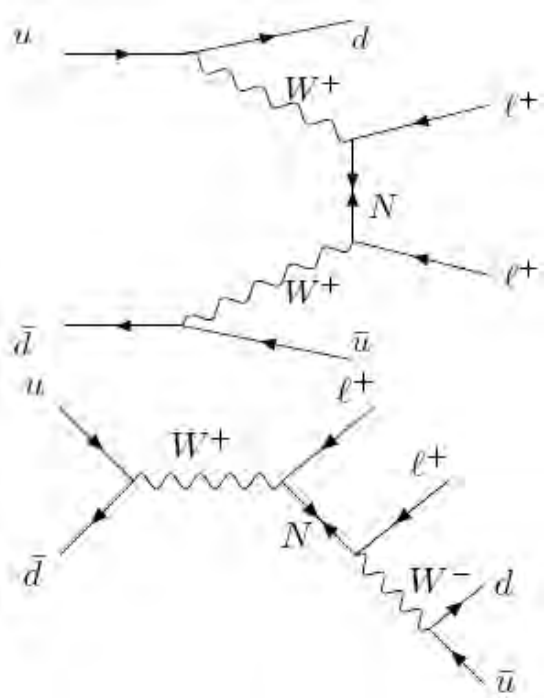
- Two jets preferable
- Color enhancements
- $\gamma\gamma$ collider diagrams?

Assume same parameters as ILC $\Rightarrow$ similar cross-sections
$\Lambda > Q \sim \mathrm{TeV}$ and $L = 100\ \mathrm{fb}^{-1}$

- *qg* and *gg* events dominate *qq* at LHC energies
- Probe different coupling constants: light quark/heavy lepton.
- Higher possibility of resonant production

Tevatron has little hope of LNV discovery from this perspective!

Hadron collider search backgrounds



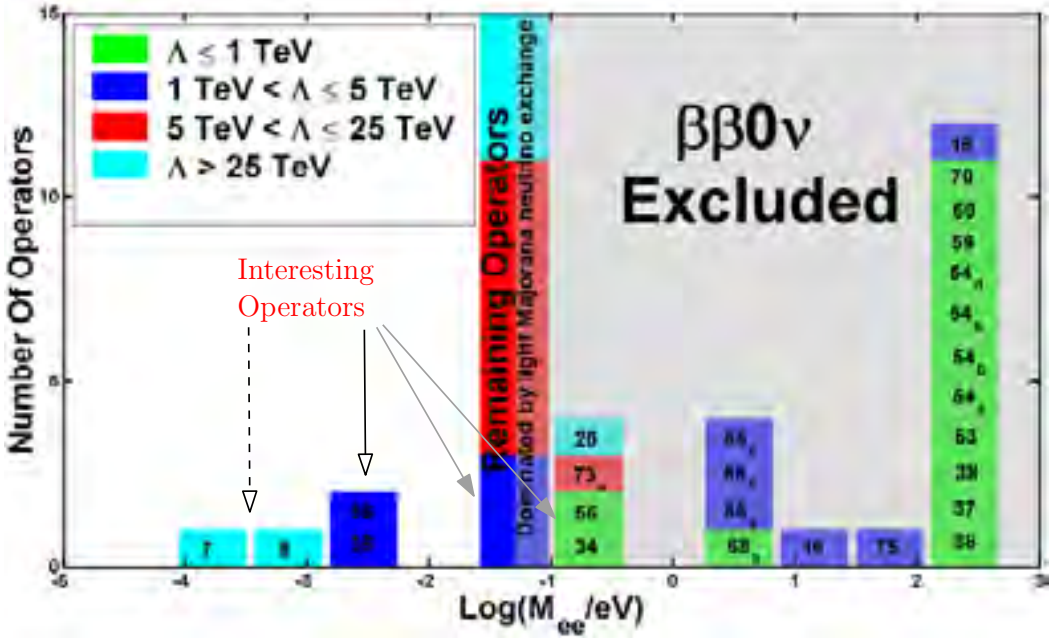
Dilepton signal + jets and no missing energy!

- Ideally no SM background
- Real world is not ideal!
  - Processes with missing energy
  - Missed final state leptons
  - Misidentified lepton charge: $\mu^+ \leftrightarrow \mu^-$
- Reduce Background... no jet kinematic cuts!
  - Well isolated final state leptons
  - Look for μe final state events

Luckily, experimentalists have vast experience searching for this signal in simple SUSY scenarios!
Distinguishing LNV from SUSY at LHC is difficult

Suggestions for future study

Bottom up model building of “interesting” operators



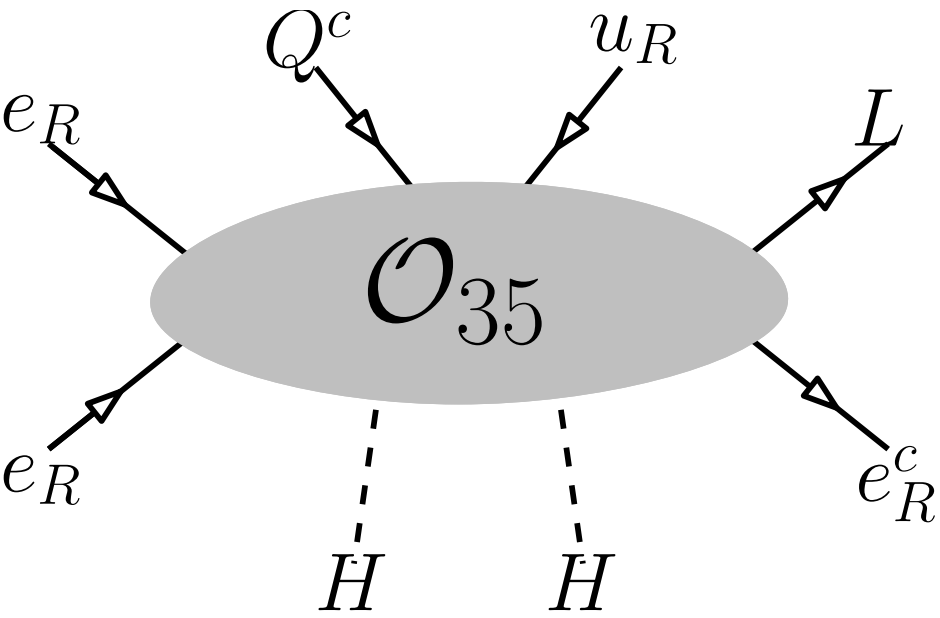
Possible new directions:

- Expand vertices-reveal model
- Detailed LHC/ILC analysis
  - PDF, flux, background...
  - Include resonance
- Combine operators
- Cosmological implications

We isolate a number of TeV scale, unconstrained operators that can naturally generate small neutrino masses.

What is the nature of such models? Can they do anything else for us?

Model building suggestions: $\mathcal{O}_{35}$



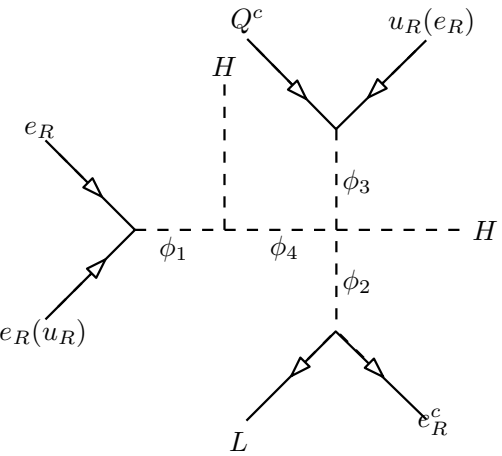
$$\frac{1}{\Lambda^7}\bar{e}^c\bar{e}^c L^i e^c \overline{Q}_j \bar{u}^c H^j H^k \epsilon_{ik}$$

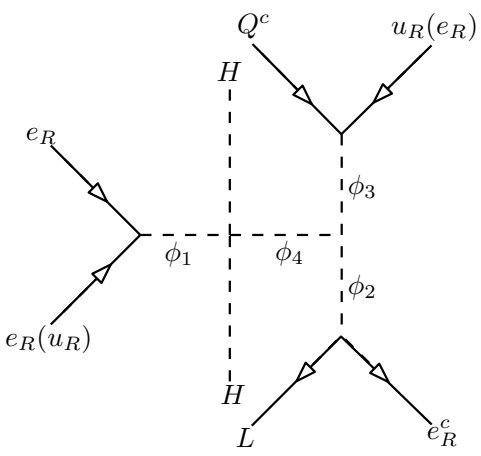
$$\Lambda \approx 2 \text{ TeV}$$

- Many models defined by number of Lorentz contractions
- Pure vector exchange is not allowed... Need scalar/tensor
- Other operators with same structure!

Model building suggestions: $\mathcal{O}_{35}$

Simple Scalar mediation cases





Model Characteristics

- 4 scalar fields needed
- Constrained couplings
- essential ϕ_4 role
- Variations
  - 2 Lorentz contractions
  - Higgs placement
- Easy to write Lagrangian!

Avoiding lower dimensional operators suggests Lagrangian constraints... Guaranteeing this in a model setting requires nontrivial symmetry structure!

Summary

We expect new physics above the TeV scale to stabilize the SM

- The categorization and exploration of effective low energy operators is a model independent probe of new physics
- I consider effective LNV operators to:
  - Categorize *all* relevant operators by scale
  - Extract current constraints
  - Predict/suggest future probes
- Analysis is useful on a number of levels!
- Future work still needed... Great prospects!

Summary

We expect new physics above the TeV scale to stabilize the SM

- The categorization and exploration of effective low energy operators is a model independent probe of new physics
- I consider effective LNV operators to:
  - Categorize *all* relevant operators by scale
  - Extract current constraints
  - Predict/suggest future probes
- Analysis is useful on a number of levels!
- Future work still needed... Great prospects!

This illustrates the need for complementary experimental searches to truly understand LNV.

Begin Backup Slides

Rare gauge boson decays

Gauge bosons can couple to operators in three ways

1. Close golden channel quark fermion loops
2. Couple to external fermion legs
3. Explicit goldstone Higgs operator content

- Z boson decay... $Z \to \ell_\alpha \ell_\beta + 4j$ (allowed)
  - Invisible Z-decay width: $Z \to \nu_\alpha \nu_\beta$ (close 2 loops)

$$\Gamma_{inv}^{exp} < 2\mathrm{MeV} \quad \Rightarrow \mathcal{A} < \sqrt{4\pi \frac{2\mathrm{MeV}}{\mathrm{M_Z}}} \Rightarrow \mathrm{y}^2 \left(\frac{\mathrm{v}}{\tilde{\ }}\right)^3 < 4 \times 10^2$$

- W decay is similar... $W \to \ell_\alpha \ell_\beta + 2j$ (loop suppressed!)

$$\Gamma_{LNV} \leq m_W \frac{4\pi}{(16\pi^2)^5} \left(\frac{v}{\Lambda}\right)^{10} \approx 10^{-5}\ \mathrm{MeV}$$

Implied neutrino mass textures

$$m_{ee} = m_1 \cos^2\theta_{12}\cos^2\theta_{13} + m_2 \sin^2\theta_{12}\cos^2\theta_{13} e^{2\imath\phi_2} + m_3 \sin^2\theta_{13} e^{2\imath(\phi_3-\delta)}$$

For the normal hierarchy, $m_2 = \sqrt{m_1^2 + \Delta_S^2}$, $m_3 = \sqrt{m_1^2 + \Delta_S^2 + \Delta_A^2}$

Solve for m_1 assuming small m_{ee}, θ_{13}, and $\eta = \sqrt{\frac{\Delta_S^2}{\Delta_A^2}}$

$$\begin{aligned}
\frac{m_1}{\sqrt{\Delta m_{12}^2}} &\approx \left(m_1^{(0)} + \delta m(m_{ee}, \phi_2)\right) / \sqrt{\Delta m_{12}^2} \\
&= \frac{\sin^2\theta_S}{\cos^{1/2} 2\theta_S} + \frac{\cos^2\theta_S}{\cos 2\theta_S}\left(\frac{m_{ee}}{\sqrt{\Delta_S^2}} - \frac{\theta_{13}^2}{\eta} e^{2\imath(\phi_3-\delta)} + \imath\,(2\phi_2 - \pi)\,\frac{\sin^2 2\theta_S}{4\cos^{1/2} 2\theta_S}\right).
\end{aligned}$$

In a similar way, expand m_2 and m_3

Now look at the magnitude of $m_{\alpha\beta}$ and compare with textures

Implied neutrino mass textures

Analytic results

$$\frac{m^2_{\alpha\beta}}{\Delta^2_A} = .25\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + .31\eta\cos 2\phi_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$+ .28\eta^2 \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1.45 & 1.45 \\ 1 & 1.45 & 1.45 \end{pmatrix} + .50\theta_{13}^2 \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$+ .75\eta\theta_{13}\cos(2\phi_3 - \delta) \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} + .19\eta(2\phi_2 - \pi)\sin 2\phi_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

- Zeroth order
- First order
- Second order

$$M \propto \begin{pmatrix} \lambda_e & \lambda_\mu & \lambda_\tau \\ \lambda_\mu & \lambda_\mu & \lambda_\tau \\ \lambda_\tau & \lambda_\tau & \lambda_\tau \end{pmatrix} \quad , \quad M \propto \begin{pmatrix} \lambda_e\lambda_e & \lambda_e\lambda_\mu & \lambda_e\lambda_\tau \\ \lambda_e\lambda_\mu & \lambda_\mu\lambda_\mu & \lambda_\mu\lambda_\tau \\ \lambda_e\lambda_\tau & \lambda_\mu\lambda_\tau & \lambda_\tau\lambda_\tau \end{pmatrix}$$

These do not fit... But what about parameter uncertainties?