Thermonuclear Reaction Rates in the Sun and Stars

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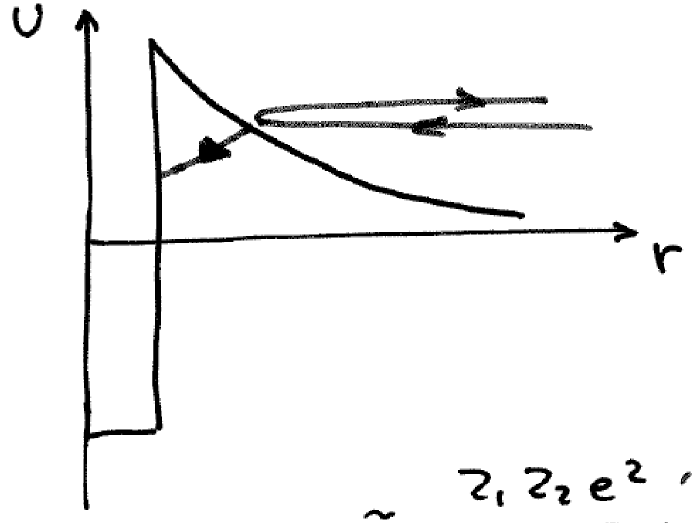
Conclusions:

- The best available calculation of theoretical reaction rates in the Sun is AG, Bahcall (1998)
- This calculation is not good enough: 1. few % accuracy is not better than neutrinos and helioseismology, 2. the quoted few % is not a rigorous number, it comes from the ceiling, 3. this is an "engineering calculation"
- There must be a way to do it better

Solar and solar neutrino models

- To predict the structure of the Sun and neutrino emission need to know nuclear reaction rates
- T ~ 1keV, hard to measure, special experiments, extrapolations, give raw rates, (the rates for an ideal gas)
- Solar plasma in the core is not an ideal gas

2-body: p ⊕ ⟶   ⟵ ⊕ p



in plasma:

$$\frac{1}{r} \to \frac{1}{r} e^{-r/R_D}$$

$$U = \frac{Z_1 Z_2 e^2}{r} e^{-\frac{r}{R_D}} \approx$$

$$\approx \frac{Z_1 Z_2 e^2}{r} - \frac{Z_1 Z_2 e^2}{R_D} \Rightarrow$$

$$w = w_0 e^{\Lambda} \qquad \boxed{\Lambda \equiv \frac{Z_1 Z_2 e^2}{T R_D}}$$

→ 5%
→ 20%
→ 40%

need ~ 1%

Λ shows how ideal is the plasma

Salpeter screening formula

2. ENHANCEMENT OF FUSION RATES

The solar core plasma is dense enough that it noticeably enhances fusion rates as compared to the rates in a rarefied plasma of the same temperature. As explained by Salpeter (1954), the rate of a fusion of two nuclei of charges Z_1 and Z_2 is increased by a factor

$$f = \exp \Lambda \,, \tag{1}$$

where

$$\Lambda = Z_1 Z_2 \frac{e^2}{T R_D} \,. \tag{2}$$

Here R_D is the Debye radius,

$$\frac{1}{R_D^2} = 4\pi \beta n e^2 \zeta^2 \,, \tag{3}$$

with

$$\zeta = \left[\Sigma_i X_i \frac{Z_i^2}{A_i} + \left(\frac{f'}{f} \right) \Sigma_i X_i \frac{Z_i}{A_i} \right]^{1/2} . \tag{4}$$

Here $\beta = 1/T$; n is the baryon density; X_i, Z_i, and A_i are, respectively, the mass fraction, the nuclear charge, and the atomic weight of ions of type i. The quantity $f'/f \simeq 0.92$ accounts for electron degeneracy. Equation (4) is the same

This is correct to about $\sim \Lambda^2$

“solving solar neutrinos”

- many attempts to solve solar neutrinos by changing reaction rates in order Λ , that is showing that Salpeter formula is wrong
- all these are wrong
- show it to learn the physics of screening, then try to do it right

Easy ones:

(from Bahcall, Brown, AG, Sawyer 2002)

Many claims that Gibbs is wrong: non-equilibrium, "Tsallis statistics", etc.

$$\delta = \frac{\tau_{\rm Coulomb}}{\tau_{\rm nuclear}} = 10^{-28}\left[\left(\frac{\tau_{\rm nuclear}}{10^{10}\,{\rm yr}}\right) \times \left(\frac{20\,{\rm keV}}{E}\right)^{3/2}\left(\frac{\rho}{150\,{\rm g\,cm^{-3}}}\right)\right]^{-1}.$$

Innocent until proven guilty

cloud-cloud interaction

force is derivative of energy, but.....

Suppression of thermonuclear reactions in dense plasmas instead of Salpeter's enhancement

solar interior it is found that the decrease approaches a factor $1/2$ for reactions with Be nuclei, and this could be relevant for the problem of solar neutrino deficit.

$$\Lambda_{ij} = -\frac{e^2}{2\sqrt{\pi}Td}\int_{-1}^{1}dx\int_{-1}^{1}dz\int_{0}^{\infty}dyy^2\exp(-y^2) \times \left\{ Z_i^2 \frac{\sum_\alpha \frac{1}{d_\alpha^2}(2s_{\alpha,i}^2 W(s_{\alpha,i})+1)}{\sqrt{\left(\sum_\alpha \frac{1}{d_\alpha^2}W(s_{\alpha,i})\right)\left(\sum_\alpha \frac{1}{d_\alpha^2}\right)}} + Z_j^2 \frac{\sum_\alpha \frac{1}{d_\alpha^2}(2s_{\alpha,j}^2 W(s_{\alpha,j})+1)}{\sqrt{\left(\sum_\alpha \frac{1}{d_\alpha^2}W(s_{\alpha,j})\right)\left(\sum_\alpha \frac{1}{d_\alpha^2}\right)}} \right\} \quad (11)$$

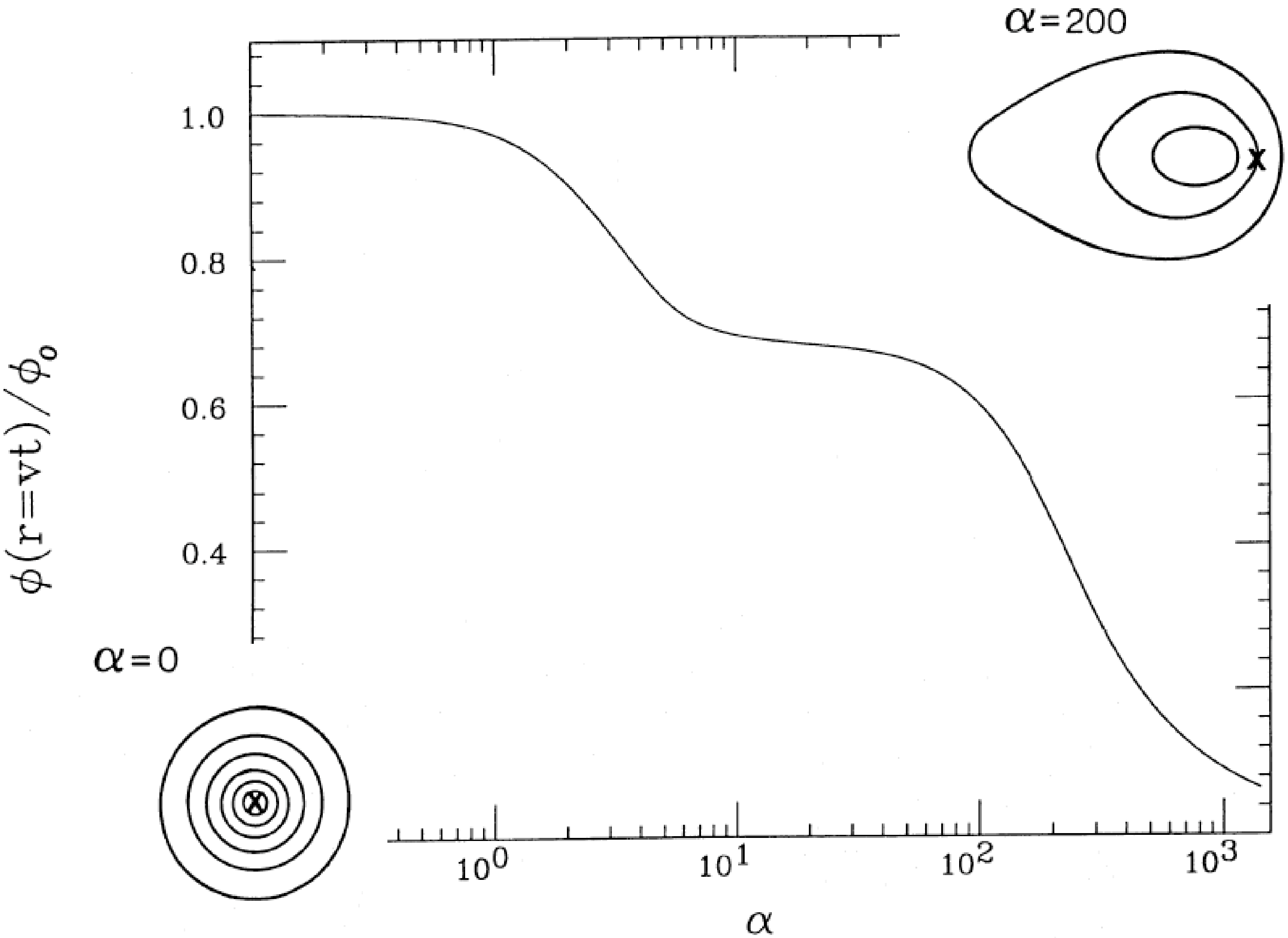
where the sum over α includes both electrons and all ion species of the plasma, $\alpha = \{e, i..j..\}$; $W(s) = 1 + s\exp(-s^2)\left(i\sqrt{\pi} - 2\int_0^s \exp(t^2)dt\right)$ is the

$$f_0(\text{Salpeter}) = 1 + gZ_1Z_2$$

$$1 - g_1Z_1^2 - g_2Z_2^2,$$

Dynamic Screening

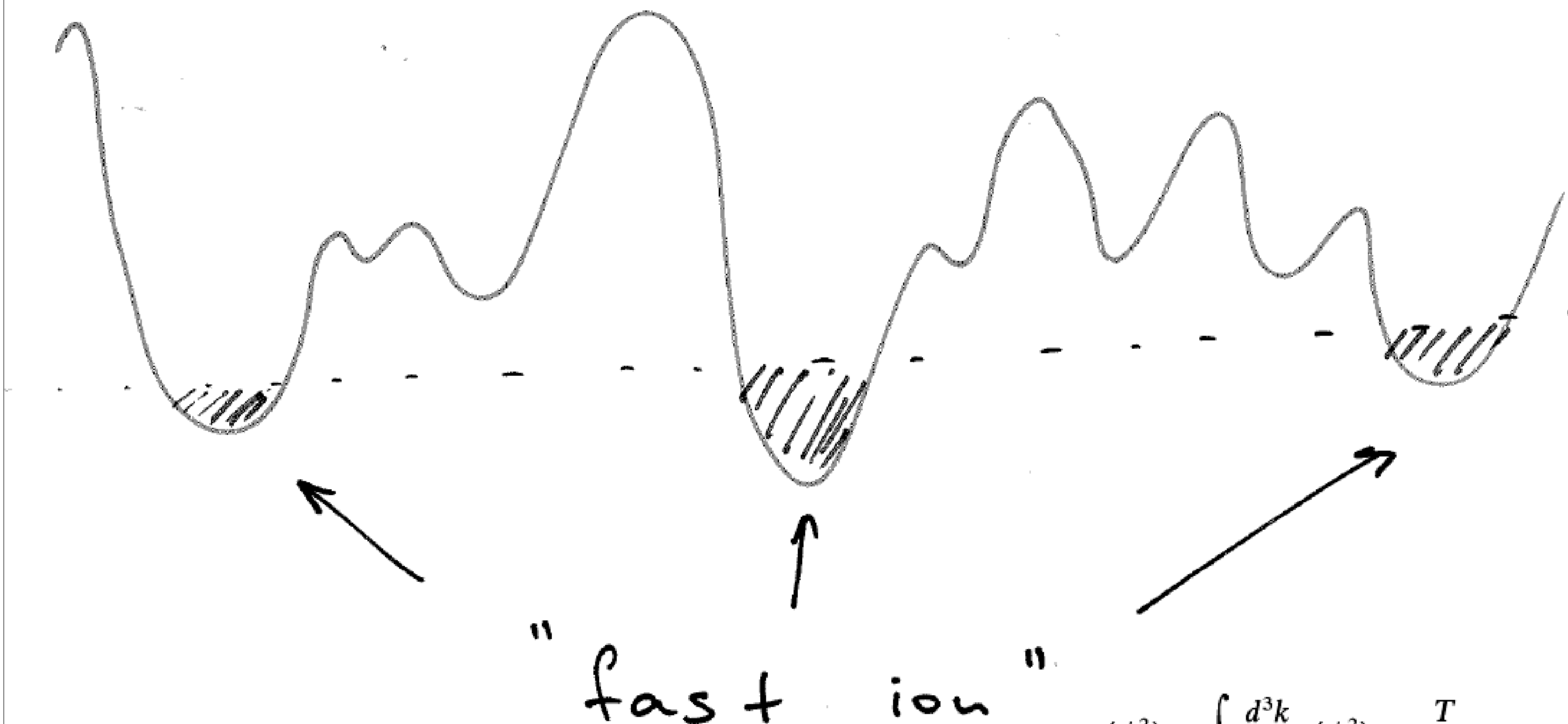
- Good interesting paper, which even Salpeter believed to be correct
- Brown, Sawyer (1997), AG (1997) showed it wrong



IG. 2.—Velocity dependence of the polarization potential at a moving ^{4}He nucleus in the solar core. The plateau around $\alpha = 10$ corresponds to electron ening only.

$$n_{1,2}(r) = C_{1,2}\,\exp\left[-\beta Z_{1,2}\, e\phi(r)\right] .$$

$$R = K\langle n_1(r) n_2(r)\rangle$$
$$= K C_1\, C_2\left[1 + \tfrac{1}{2}\beta^2 e^2 (Z_1 + Z_2)^2 \langle \phi^2\rangle\right] ,$$



$$\langle \phi^2\rangle = \int \frac{d^3k}{(2\pi)^3}\,\langle \phi^2\rangle_k = \frac{T}{R_{\mathrm{D}}} .$$

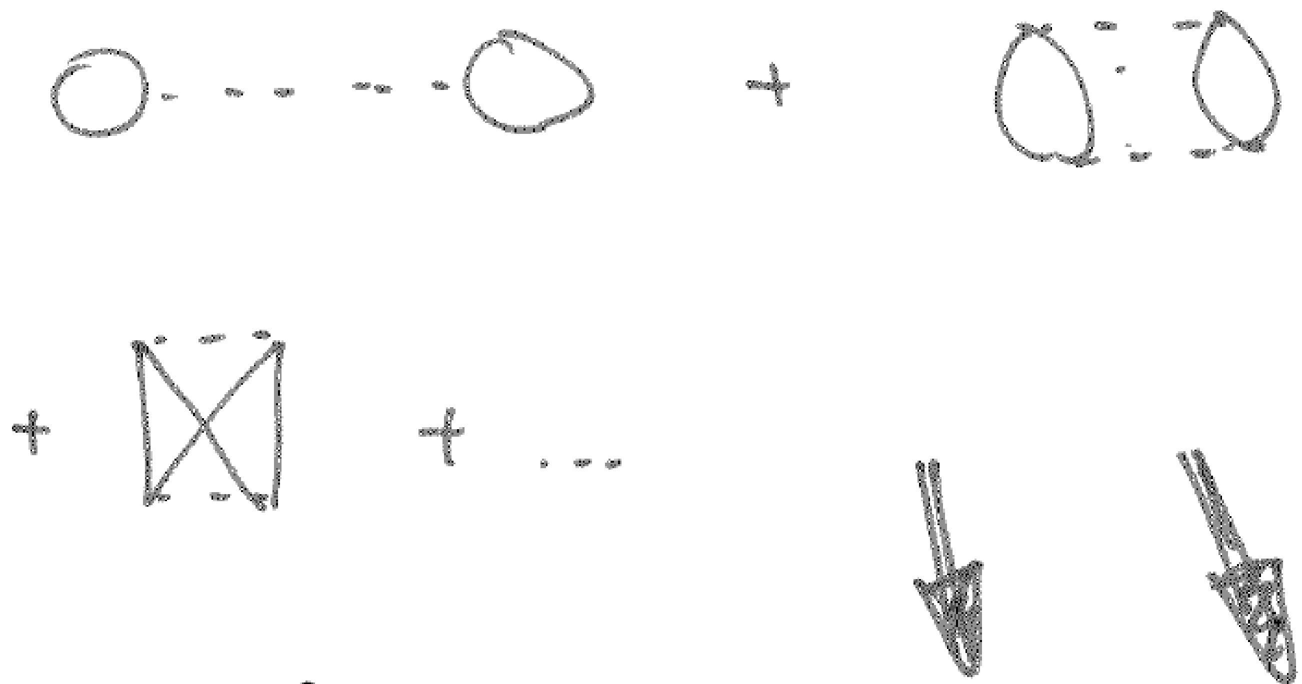
$$w = 1 + \beta^2 e^2 Z_1\, Z_2 \langle \phi^2\rangle$$

$$w = 1 + \frac{Z_1 Z_2\, e^2}{T R_{\mathrm{D}}} .$$

Correct calculations which go beyond Salpeter

- Brown, Sawyer (1998)
- AG, Bahcall (1998)
- correct, but not accurate enough

Vedenov & Larkin (1958)



Δ free energy $= - \ldots \Lambda + \ldots \Lambda^2$

- Brown, Sawyer (1998) give similar epansion for reaction rates
- not to Λ^2
- there is 1/2 He in Debye sphere
- divergent asymptotic expansions are not reliable

Engineering calculation

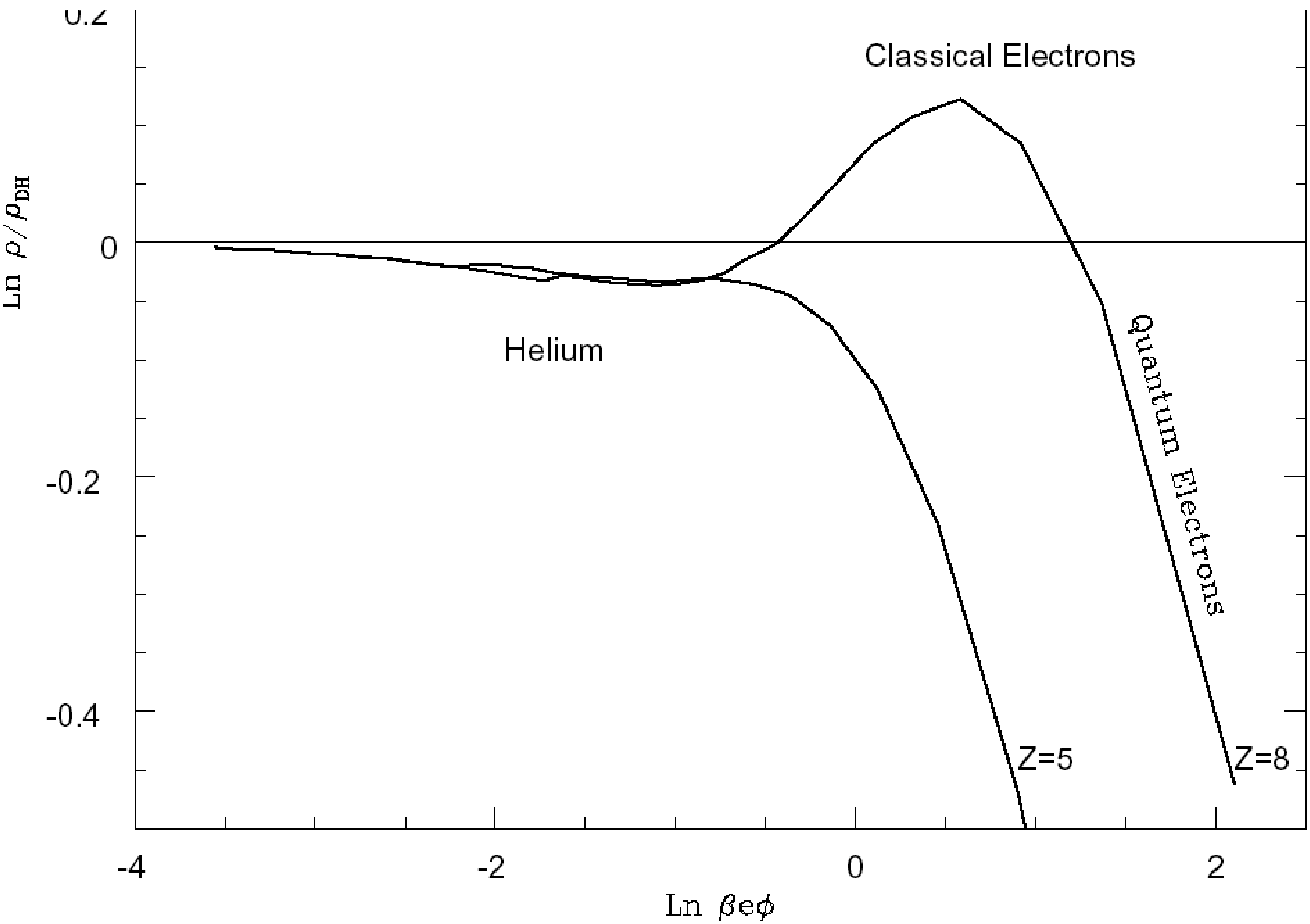
AG, Bahcall (1998)

$$\nabla^2\phi = 4\pi n\left[\left(1-\frac{Y}{2}\right)e^{\beta\phi} - (1-Y)e^{-\beta\phi} - \frac{Y}{2}\,e^{-2\beta\phi}\right],$$

But:

$$\partial_\beta\,\rho = [\tfrac{1}{2}\nabla^2 + \phi(r)]\rho\ ,$$

Monte-Carlo ions



0.2
0
-0.2
-0.4
Ln ρ/ρ_{DH}
Classical Electrons
Helium
Quantum Electrons
Z=5
Z=8
-4
-2
0
2
Ln $\beta e\phi$

ELECTROSTATIC, KINETIC, AND FREE ENERGY CORRECTIONS (%)

PARAMETER	Z					
	1	2	4	5	7	8
$\beta\,\delta U$	0.34	1.6	6.4	9.2	11.2	7.6
$\beta\,\delta F_U$	0.1	0.6	2.7	3.9	5.7	5.2
$\beta\,\delta K$	0.22	0.57	1.9	3.2	8.1	12.6
$\beta\,\delta F_K$	0.1	0.3	0.8	1.3	2.9	4.4
$\beta\,\delta F$	0.2	0.9	3.5	5.2	8.6	9.6

$r \gg \beta^{1/2}$: HIGH-TEMPERATURE EXPANSION

$$\delta K = \tfrac{1}{24} n_e \beta^2 \int 4\pi r^2 \, dr e^{-\beta V} V'^2$$

$r \ll R_D$: HYDROGENIC DENSITY MATRIX

At distances from the screened nucleus $r \ll R_D$, the potential energy is

$$V = -\frac{Z}{r} \exp\left(-\frac{r}{R_D}\right) \approx -\frac{Z}{r} + \frac{Z}{R_D} . \tag{A7}$$

The only effect of the constant correction Z/R_D is to lower electron density by the Boltzmann factor $e^{-\beta Z/R_D}$. The density matrix in the Coulomb potential can be obtained from hydrogenic eigenstates.

The kinetic energy correction is

$$\delta K = n_e e^{-\beta Z/R_D} (2\pi\beta)^{3/2} \int d^3 r [-\partial_\beta \rho - (\tfrac{3}{2}\beta^{-1} + V)\rho] . \tag{A8}$$

The diagonal of the density matrix is

$$\rho(r, \beta) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \left[\sum_{n=1}^{\infty} |R_{nl}(r)|^2 e^{\beta/2n^2} + \int_0^{\infty} \frac{dk}{2\pi} |R_{kl}(r)|^2 e^{-\beta k^2/2} \right] . \tag{A9}$$

Here the bound states of hydrogen are (e.g., Landau & Lifshitz 1977)

$$R_{nl}(r) = \frac{2}{n^{l+2}(2l+1)!} \left[\frac{(n+l)!}{(n-l-1)!} \right]^{1/2} (2r)^l e^{-r/n} F\left(-n+l+1, 2l+2, \frac{2r}{n}\right) , \tag{A10}$$

where F is the confluent hypergeometric function. The continuum states are

$$R_{kl}(r) = 2k e^{\pi/2k} \left| \Gamma\left(l+1-\frac{i}{k}\right) \right| (2kr)^l e^{-ikr} F\left(\frac{i}{k}+l+1, 2l+2, 2ikr\right) , \tag{A11}$$

Reaction Rate Corrections (%)

Reaction (1)	GB (2)	GDGC (3)	SVH (4)	DTDL (5)
$p + p$	0.5	0.0	0.5	0.2
$^{3}He + ^{4}He$......	1.7	8.2	2.4	1.8
$p + ^{7}Be$	1.5	8.5	2.6	2.3
$p + ^{14}N$	0.8	15.2	6.3	6.3

Better way

- small number of particles in Debye sphere is good for Monte-Carlo
- need a way to go from Monte-Carlo to linear screening
- electron degeneracy already included with sufficient accuracy
- need a way to Monte-Carlo quantum electrons

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