Efficient Computation of Minimum Exposure Paths in a Sensor Network Field

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Abstract. The exposure of a path p is a measure of the likelihood that an object traveling along p is detected by a network of sensors and it is formally defined as an integral over all points x of p of the sensibility (the strength of the signal coming from x) times the element of path length. The minimum exposure path (MEP) problem is, given a pair of points x and y inside a sensor field, find a path between x and y of a minimum exposure. In this paper we introduce the first rigorous treatment of the problem, designing an approximation algorithm for the MEP problem with guaranteed performance characteristics. Given a convex polygon P of size n with $O(n)$ sensors inside it and any real number $\varepsilon > 0$, our algorithm finds a path in P whose exposure is within an $1 + \varepsilon$ factor of the exposure of the MEP, in time $O(n/\varepsilon^2\psi)$, where ψ is a topological characteristic of the field. We also describe a framework for a faster implementation of our algorithm, which reduces the time by a factor of approximately $\Theta(1/\varepsilon)$, by keeping the same approximation ratio.

1 Introduction

Wireless sensor networks have been attracting the interest of computer scientists and engineers recently due to their potential to impact our everyday lives and because of their numerous applications in areas such as health care, national security, inventory tracking, surveillance, and environmental monitoring.

One of the fundamental issues in sensor networks is related to analyzing the *coverage*, or how well a network of sensors monitors the physical space for an intrusion. The coverage is a measure of the quality of service (QoS) of the sensing function and has been studied by several authors (see [5] for a recent survey of results). The concept of coverage was introduced by Gage [7], who studied it in relation to multi-robot systems. He defined three classes of coverage problems: *blanket coverage* (also known as *area coverage*), where the goal is to achieve a static arrangement of sensing elements that maximizes the detection rate of targets appearing in the region, *sweep coverage*, where the goal is to move a number of sensors across the region as to maximize the probability of detecting a target, and *barrier coverage*, where the objective is to protect the region from undetected penetration.

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The first model proposed for the barrier coverage problem is due to Meguerdichian et al. [12], who defined the *maximum breach path problem* as a problem of the following type: given a sensor field with known locations of the sensors, find a path such that the distance from any point on the path to the closest sensor is maximized. Meguerdichian et al. solve the maximum breach path problem by using the fact that there is always a maximum breach path that goes along the edges of the Voronoi diagram [4] computed for the set of sensor locations. This concept was further developed by Meguerdichian et al. [11] and by Veltri et al. [14], who define the *exposure* of a path p as an integral over all points x of p of the ability of sensing (detecting) x, which ability is given as a function that depends on the distance between x and the closest sensor, as well as the sensing time.

The *minimum exposure path* (MEP) problem is, given a sensor field and a pair of points x and y inside it, find a path between x and y of a minimum exposure. Meguerdichian et al. [11] give an exact formula for computing the MEP between two points at equal distances from the sensor in a single-sensor field and a sensing function $\gamma/d(s,x)$, where γ is a constant and $d(s,x)$ is the Euclidean distance between the sensor location s and the point x on the path. Although there is no formal proof published, the MEP problem is believed to be unsolvable in the case of multi-sensor fields. In order to solve it approximately, in [11] the region is covered by an $k \times k$ grid of points, each point is connected by edges to l other points following a given pattern, edge weights are assigned equal to the approximated exposures computed using numerical integration techniques, and finally a single pair shortest path problem is solved on the resulting graph. There is no analysis in [11] of how closely the path constructed by their algorithm approximates the MEP or of the time complexity of the algorithm, although it is easy to see that the size of the graph they construct is $\Omega(k^2 l)$ and that it is dependent on the area of the region and is independent of the number of the sensors. Veltri et al. [14] find a partial solution to the problem of exactly computing the exposure between two points in a single-sensor field and describe, for the case of many-sensor fields, a heuristic message-passing distributed algorithm that allows the sensor network to estimate a minimum exposure without knowing the network topology. Distributed algorithms for coverage problems were also studied by Li et al. in [10], for the problem of finding a path with maximum observability, and by Huang et al. in [9], who consider a dynamic version of the maximum breach and maximum observability path problems, where the topology of the sensor network may change due to new sensors being inserted, relocated, or deleted from the network.

In this paper we describe an approximation algorithm for solving the MEP problem. Our algorithm takes as input a description of a sensor field consisting of n sensors positioned inside an $O(n)$ vertex simple polygon P, two points x and y in P, and any number $\varepsilon > 0$. It returns a path between x and y in P whose exposure is within $1+\varepsilon$ factor of the exposure of the MEP. The algorithm is based on an analysis of the properties of MEPs and on a discretization of the region by constructing a Voronoi diagram and defining additional points and edges. Then a shortest path problem is solved on the resulting graph and the resulting shortest path is used as an approximation to the MEP. The time of the algorithm is $O(n/\varepsilon^2\psi)$, where ψ is a topological characteristic of the field. We also describe a faster version of the algorithm that improves the computation time by a factor of roughly $\Theta(1/\varepsilon)$.

The main contributions of this paper are the following: (i) We find an *exact* solution for the MEP problem in a single-sensor field – the previous solution [14] was valid only in special cases; (ii) We develop the first approximation algorithm for the MEP problem in a multi-sensor field with theoretically guaranteed running time and approximation ratio; (iii) We develop a theoretical framework that can be applied for designing approximation algorithms for related minimum exposure and coverage problems; (iv) Our algorithm is much faster and uses much less memory than the previous algorithms [11], [14] since the latter create a 2-D mesh of points covering the entire region, while we only place additional points on the edges.

The paper is organized as follows. In Section 2 we formally introduce the MEP problem and give some definition. In Section 3 we study MEPs in sensor fields of a single sensor and in Section 4 we study the multiple-sensor case. In Section 5 we describe our approximation algorithms for computing a minimum exposure path and in the last section we conclude with a list of open problems and ongoing work.

2 Preliminaries and Problem Formulation

We consider a connected region P in the plane of *bounded aspect ratio*, e.g., such that the ratio of the square root of the area of P and the perimeter of P is bounded. We have n identical sensors located at points $l_1, \ldots, l_n$ in P monitoring for a target in P. Each target emits a signal that the sensors try to detect. The strength of that signal diminishes with the distance traveled. The probability that a target will be detected depends also on parameters such as the energy emitted by the target, the nature of the signal, the sensitivity of the sensors, and the noise in the environment. Adopting a widely used sensibility model [8,11,14,6], we assume that the signal energy of a target at point x detected by a single sensor at point l is

$$S(l,x) = \frac{\gamma}{d(l,x)^{\mu}}\,, \tag{1}$$

where $d(l,x)$ is the Euclidian distance between l and x and γ and μ are constants. Depending on the technology and the environment, the value of μ, called *sensibility exponent*, is typically between 1 and 5.

A *sensor field* F is defined as a 3-tuple $F = (P, L, S)$, where P is a connected region in the plane, $L = \{l_1, \ldots, l_n\}$ is the set of sensor locations, and the function S, called *sensibility* of F, is defined by (1).

For the case of multiple sensors, the notion of *sensor field intensity* for a given point x in the sensor field F has been introduced in [11] in order to measure the likelihood that a target on x will be detected by any of the sensors. There are two basic variations of the model. In the *all-sensor intensity* model, the *intensity* at point x, denoted by $I_A(F, x)$, is defined as a sum of the sensibilities of individual sensors, e.g., $I_A(F, x) = \sum_{i=1}^{n} S(l_i, x)$. The all sensor intensity model reflects more accurately the capability of the sensors to detect a target, but it has also a number of weaknesses: (i) it assumes that all sensors are active during most of the time, which would be energy inefficient; (ii) it presents greater communication and data fusion challenges; (iii) the collection of data from weak sources increases the total noise-to-signal ratio.

In the *closest-sensor field intensity* model the *intensity* at a point x, denoted by $I_C(F, x)$, is defined as $I_C(F, x) = S(l_i, x)$, where l_i is the closest sensor to x.

Let p be a path given as $p = \{p(t) \in P \mid t \in [t_1, t_2]\}$, where $[t_1, t_2]$ is a given interval and $p(t)$ is a continuous function differentiable everywhere in $[t_1, t_2]$ except for a finite number of point. The *exposure* of p with regard to intensity model I and field F is defined [14,11] as

$$\exp(p, I, F) = \int_{t_1}^{t_2} I(F, p(t)) \left| \frac{\mathrm{d}p(t)}{\mathrm{d}t} \right| \mathrm{d}t\,, \tag{2}$$

where $I(F, x)$ is either $I_A(F, x)$ or $I_C(F, x)$ and $|\mathrm{d}p(t)/\mathrm{d}t|$ is the element of arc length. In the rest of this paper we assume that $I(F, x) = I_C(F, x)$. The definition of exposure accounts for the fact that the probability for a target traveling at a constant speed along the path p to be detected by a sensor is proportional to the intensity of the field along p and the length of the path. A *minimum exposure path* $\mathrm{MEP}(x, y, F)$ between x and y is defined as a path between x and y in P with a minimum exposure.

The *minimum exposure path problem* is, given a sensor field $F = (P, L, S)$ and a pair of points x and y, find $\mathrm{MEP}(x, y, F)$. In order to simplify the notations, we use $\exp(p)$ or $\exp(p, S)$ instead of $\exp(p, I, F)$ and $\mathrm{MEP}(x, y)$ instead of $\mathrm{MEP}(x, y, F)$, when P, L, F, and/or I are clear from the context.

We end this section with several definitions from graph theory. A *graph* G is a pair of two sets denoted by $V(G)$ and $E(G)$, where $V(G)$ is the set of the *vertices* and $E(G)$ is the set of the *edges* of G, where each edge is a pair (v, w) of vertices. A *path* p in G is a sequence $v_0, \ldots, v_k$ of vertices, where $(v_{i-1}, v_i) \in E(G)$ for $i = 1, \ldots, k$. If $k = 0$ then p is a *null* path. If there are weights associated with the edges of G, then the *length* of p is defined as the sum of the weights of all edges (v_{i-1}, v_i). Given two vertices $v, w \in V(G)$, the *distance* between v and w is the minimum length of any path between v and w (infinity, if there is no such path). The *shortest path problem* is, given v and w, find a shortest path between v and w.

3 Single-Sensor Fields

Next, we will study the MEP problem in the case of a single sensor. Without loss of generality, in the rest of this paper we assume that $\gamma = 1$, where γ is the constant from (1). (Changing γ scales the exposures of all paths by the same factor and hence preserves the minimum exposure paths.)

Case A: Unrestricted region. We will start by considering the case of an unbounded region, e.g., where P is the entire plane. We use polar coordinates to represent each point q as a pair (ρ, α), where ρ is the distance between q and the origin O (which we choose to be the sensor location) and $\alpha \in [0, 2\pi)$ is the angle between the polar axis and $\overrightarrow{Oq}$. The exposure of a path p with endpoints $x(\rho_0, 0)$ and $y(\rho_\alpha, \alpha)$ in polar coordinates given as $p = \{(\rho(\theta), \theta) \mid \theta \in [0, \alpha]\}$ can be written as

$$\exp(p, d^{-\mu-1}) = \int_0^\alpha \rho(\theta)^{-\mu-1} \sqrt{\rho(\theta)^2 + \rho'(\theta)^2}\, \mathrm{d}\theta. \tag{3}$$

Using the *Beltrami identity* [15], we can find that if ρ is a nonnegative function defined in the interval $[0, \alpha]$ that minimizes the integral (3), then

$$\rho(\theta) = \begin{cases} \rho_0 (\dfrac{\rho_\alpha}{\rho_0})^{\frac{\theta}{\alpha}} & \text{if } \mu = 0; \quad (4) \\ \left(\dfrac{\rho_0^\mu \sin(\mu\alpha - \mu\theta) + \rho_\alpha^\mu \sin(\mu\theta)}{\sin(\mu\alpha)} \right)^{1/\mu} & \text{if } \mu \neq 0. \quad (5) \end{cases}$$

Formulas (4)–(5) were derived in [14] using the Euler-Lagrange differential equation, but were not analyzed whether they correspond to a minimum of (3). But since (4)-(5) are only necessary conditions, one needs to additionally check whether a function ρ satisfying (4) or (5) for a particular set of values for μ, ρ_0, ρ_α, and α is a minimum or an inflexion point. (Clearly, ρ from (4) or (5) can not be a maximum since (3) is unbounded from above for $\mu \geq 0$ – it tends to infinity when $\rho \to 0$.) Consider the following two cases.
Case 1: $\mu = 0$. Let ϕ_M be the set of all nonnegative continuous functions defined in $(0, \alpha]$ and upper bounded by M. The integral (3) is unbounded from above (for any $\mu \leq 0$) as it tends to infinity when $\rho \to \infty$. Therefore, for some M sufficiently large, the exact lower bound of (3) for all functions in $(0, \alpha]$ will be the same as the exact lower bound of (3) restricted to the set of functions from ϕ_M. But ϕ_M is a compact set and, hence, the exact lower bound over ϕ_M (and therefore over all functions in $(0, \alpha]$) is reached for some function $\widetilde{\rho}$ from ϕ_M. Since there is a unique function satisfying the necessary condition (4), then $\widetilde{\rho}$ should be the function defined by (4). Hence (4) does define a MEP between x and y (it is not an inflexion point).

In order to compute the exposure of that path, substitute the expressions for ρ from (4) into the exposure expression (3), resulting in

$$\operatorname{minExp}(x, y, d^{-1}) = \int_0^\alpha \sqrt{1 + \frac{\ln^2(\rho_\alpha/\rho_0)}{\alpha^2}}\, \mathrm{d}\theta = \sqrt{\alpha^2 + \ln^2(\rho_\alpha/\rho_0)}\, . \tag{6}$$

Case 2: $\mu > 0$. In this case the function (5) may or may not represent a minimum, depending on the values of α, ρ_0, and ρ_α. For instance, if $\alpha = \pi/\mu$, the path

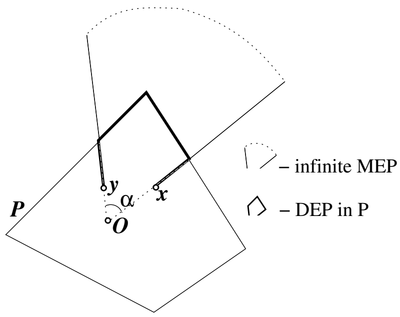

Fig. 1. A MEP in an infinite region may be infinite. In a polygon, such a path is projected into a DEP.

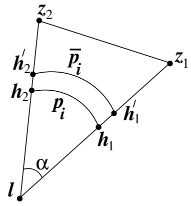

Fig. 2. Illustration to the proof of Lemma 7

given by (5) is not defined. If $\rho_0 = \rho_a$ and $\alpha \to (\pi/\mu)^+$, ρ is unbounded from above and hence ρ would not define an optimal path for α close enough to π/μ. If $\rho_0 = \rho_a$ and $\alpha \to (\pi/\mu)^-$, ρ is unbounded from below (and, in particular, gets negative values). In these cases (5) does not correspond to a solution of the optimization problem and the minimum of (3) is not reached for any (finite) function ρ. However, as we show next, a MEP always exists if the region P is bounded.

Case B: Minimum exposure paths in a polygonal region P**.** Intuitively, if we consider paths in the entire plane in the case where (5) corresponds to an inflexion point, the MEP from x to y will follow the ray from x in the direction $\overrightarrow{Ox}$ to infinity, then move along an infinite circle to align with the line yO (the exposure along that semicircle will be zero), and finally move in the direction of $\overrightarrow{yO}$ to point y (Figure 1). Although this path is of infinite length, its exposure if finite; the exposure of the path is $\mathrm{minExp}_1(x, y, d^{-\mu-1}) = (\rho_0^{-\mu} + \rho_\alpha^{-\mu})/\mu$.

In a polygonal region, the portion of the path described above that is outside P is replaced by the path of lower exposure among the two paths along the boundary of P connecting the same endpoints. We will refer to the latter path as the *direct escape path* (DEP). As a DEP in P is a chain of straightline segments, we will need a formula for the exposure along a single such segment. If the segment xy belongs to a line containing point O, then the exposure along the DEP p between the points $x(\rho_1, 0)$ and $y(\rho_2, 0)$, $0 < \rho_1 \leq \rho_2$, which we denote by $minExp_1$, can be computed by the formula

$$\mathrm{minExp}_1(x, y, d^{-\mu-1}) = \exp(p, d^{-\mu-1}) = \begin{cases} \ln\rho_2 - \ln\rho_1 & \text{if } \mu = 0 \quad (7) \\ \dfrac{1}{\mu}(\rho_1^{-\mu} - \rho_2^{-\mu}) & \text{if } \mu > 0. \quad (8) \end{cases}$$

Otherwise, the exposure $\mathrm{minExp}_1(x, y, d^{-\mu-1})$ along the segment (DEP) p between points $x(\rho_0, 0)$ and y such that $\angle Oyx = \pi/2$ and $\angle xOy = \alpha$ is

$$\exp(p, d^{-\mu-1}) = \begin{cases} \ln(\sec\alpha + \tan\alpha) & \text{if } \mu = 0 \\ \rho_0^{-\mu}\, {}_2F_1\left(\frac{1}{2}, \frac{\mu+1}{2}; \frac{3}{2}; -\tan^2\alpha\right)\tan(\alpha) & \text{if } \mu > 0, \end{cases} \quad (9)$$

where ${}_2F_1$ is the Gaussian hypergeometric function [1]. A segment xy for which $\angle Oyx \neq \pi/2$ can always be represented as a sum or a difference of segments of the above type.

Next, the exposure of the path p defined by (5) is

$$\text{minExp}_2(x, y, d^{-\mu-1}) = \frac{\sin(\mu\alpha)(\tan(\mu\alpha - c_2) + \tan c_2)}{\mu\sqrt{\rho_0^{2\mu} + \rho_\alpha^{2\mu} - 2\rho_0^\mu\rho_\alpha^\mu\cos(\mu\alpha)}}, \quad (10)$$

where $\tan(c_2) = \frac{\rho_\alpha^\mu - \rho_0^\mu\cos(\mu\alpha)}{\rho_0^\mu\sin(\mu\alpha)}$.

Finally, the minimum exposure between x and y is determined as

$$\text{minExp}(x, y, d^{-\mu}) = \min\{\text{minExp}_1(x, y, d^{-\mu}),\ \text{minExp}_2(x, y, d^{-\mu})\}.$$

The path corresponding to the smaller of the two exposures is the MEP.

4 Multiple-Sensor Fields

Here we consider the case of a sensor field $F = (P, L, S)$ with an arbitrary number n of sensors, where $L = \{l_1, \ldots, l_n\}$ is the set of the locations of the sensors and P is a convex polygon of size $O(n)$ that contains all points l_i.

We construct a *Voronoi diagram* Vor(L) for L in P, which is a tessellation of P into n convex polygons $C_1, \ldots, C_n$, which we call *cells* of Vor(L), such that C_i is the set of all points that are closer to l_i than to any other point from L. The Voronoi diagram can be constructed in $O(n \log n)$ time. (See [4] for more information about Voronoi diagrams.) Denote by $V(C_i)$ and $E(C_i)$ the sets of the vertices and the edges of C_i, respectively.

Next we analyze the structure of the MEP between any two points x and y from P. First we consider the case where x and y belong to the same cell C_i.

Lemma 1. *Define sensor fields $F_1 = (C_i, L, S)$ and $F_2 = (P', L, S)$, where $|L| = 1$ and P' is the entire plane and let x and y be points from C_i. Then*

(a) the minimum exposure path $p = \text{MEP}(x, y, F_1)$ either contains a point from the boundary of C_i, or $p = \text{MEP}(x, y, F_2)$;
(b) the intersection between p and any edge of P is either empty or a single segment.

Proof. (a) If p is disjoint with the boundary of C_i, then p is a stationary point for (3) and hence it can be determined by the method discussed in Section 3 Case A.

(b) Assume that claim (b) does not hold. Then there will exist a subpath p' of p that joins a pair of points x_1, x_2 on an edge of C_i and whose interior is entirely inside C_i. Then, by (a), $p' = \mathrm{MEP}(x_1, x_2, F_2)$. Consider the path p'' with endpoints x_1, x_2 that is symmetrical to p' with respect to the line x_1x_2. Then p'' will have a smaller intensity and the same element of arc length compared to p' and hence, by (2), a smaller exposure, which is a contradiction. □

Next we consider the case where x and y can belong to different cells of $\mathrm{Vor}(L)$.

Lemma 2. *Given two points $x \in C_i$ and $y \in C_j$, $i \neq j$,* $\mathrm{MEP}(x, y, F)$ *consists of a sequence of subpath each of them of one of the following types:*

(i) a MEP from x to a point on an edge of C_i or from a point on an edge of C_j to y;
(ii) a MEP between points belonging to two different edges of the same cell of $\mathrm{Vor}(L)$*;*
(iii) a segment on an edge of $\mathrm{Vor}(L)$*.*

Proof. Follows from the discussion in Section 3 and Lemma 1. □

5 An Approximation Algorithm for Constructing MEPs

Next we describe and analyze an algorithm that computes an approximation of the MEP between a pair of points x and y. In the algorithm, we first discretize the region by triangulating it and creating a set $\mathcal{S}$ of additional points called *Steiner points* (SPs) on the edges of the triangulation. This is similar to the discretization schemes used for solving shortest path problems on weighted polyhedral surfaces, e.g., [3]. Then we define a graph with a vertex set $\{x\} \cup \{y\} \cup \mathrm{Vor}(L) \cup \mathcal{S}$ and an edge between any pair of vertices belonging to the same triangle. We define a weight on each edge (v, w) equal to the exposure either along the minimum exposure path between v and w in the triangle containing v and w, if v and w belong to different edges, or along the edge containing v and w, otherwise (see Lemma 2). Then the algorithm finds the shortest path in the resulting graph between x and y using a modification of Dijkstra's algorithm. Next we describe the steps in more detail and analyze the accuracy and the efficiency of the algorithm.

5.1 Defining Steiner Points

First we will define "empty" regions around each sensor location l_i that will contain no SPs. The rationale is to limit the number of SPs we have to define in each Voronoi cell C_i, because the number of SPs needed to achieve a given approximation ratio increases when the distance to l_i decreases. We need the following properties.

Lemma 3. *Let F_1 and F_2 be fields with sensibility exponents $\mu_1 = 0$ and $\mu_2 > \mu_1$, respectively. Given two points $x_1(r, \alpha_1)$ and $x_2(r, \alpha_2)$ belonging to cell $C_i \in \mathrm{Vor}(L)$, let*

$$p_1 = \mathrm{MEP}(x_1, x_2, F_1) = \{(\rho_1(\theta), \theta) \mid \theta \in [0, \alpha]\}$$

and

$$p_2 = \mathrm{MEP}(x_1, x_2, F_2) = \{(\rho_2(\theta), \theta) \mid \theta \in [0, \alpha]\}.$$

Then $\rho_1(\theta) \leq \rho_2(\theta)$ for all $\theta \in (\alpha_1, \alpha_2)$.

Let F be a field with a scalability exponent $\mu \geq 0$, and let $d_i = \min\{d(l_i, z) \mid z \in E(C_i)\}$. Define a circle κ_i with center l_i and radius d_i.

Lemma 4. *Any MEP in F with both endpoints on $E(C_i)$ contains no points from the inside of κ_i.*

Proof. Suppose p is a MEP for a sensor field with sensibility exponent μ that contains a point from the inside of κ_i. Then p contains a subpath p_1 with endpoints, say, a and b on κ_i and the rest of p_1 in the interior of κ_i. Consider the following two cases:

(i) $\mu = 0$. By (4), the portion p_1' on κ_i between a and b, being a minimum exposure path, has a smaller exposure than p_1. Replacing p_1 by p_1' in p results in a path with a smaller exposure than p, a contradiction.

(ii) $\mu > 0$. Combine the proof of case (i) with Lemma 3. □

Next we define $\mathcal{S}$, the set of Steiner points for $\mathrm{Vor}(L)$. For each $C_i \in \mathrm{Vor}(L)$, triangulate C_i by adding straightline segments joining l_i to each vertex of $V(C_i)$. Let T_i be the resulting set of triangles for C_i, let $\mathcal{T}$ be the resulting triangulation of $\mathrm{Vor}(L)$, and let $t \in T_i$ for some i. Let l_i, a, b be the vertices of t and $d(l_i, a) \geq d(l_i, b)$. Call (l_i, a) and (l_i, b) *new* edges and call (a, b) an *old* edge. Let $l = d(l_i, a)$ and let $\varepsilon > 0$. Define a set of Steiner points $s_0, s_1, \ldots, s_\lambda$ on the segment $\overline{l_i x}$ such that

$$d(l_i, s_0) = d_i, \quad d(l_i, s_{j-1}) < d(l_i, s_j), \quad d(s_{j-1}, s_j) = \varepsilon d(l_i, s_{j-1}), \tag{11}$$

for $j = 1, \ldots, \lambda$, where λ is chosen such that $d(l_i, s_\lambda) \leq l < d(l_i, s_{\lambda+1})$. (We used Lemma 4 for justifying the definition of s_0.) In a similar way we define SPs on the segment $\overline{l_i y}$. For the segment $\overline{xy}$ we define the SPs $s_0', \ldots, s_{\lambda'}'$ such that

$$s_0' = a,\ s_{\lambda'}' = b,\ d(l_i, s_{j-1}') < d(l_i, s_j'),\ d(s_{j-1}', s_j') = \varepsilon l \text{ for } j = 1, \ldots, \lambda' - 1,$$
$$\text{and } d(s_{\lambda'-1}', s_{\lambda'}') \leq \varepsilon l.$$

Lemma 5. *The number of SPs on the segments of the triangle t is $O(\ln(l/d)/\varepsilon)$.*

Proof. From (11), $d(l_i, s_j) = (1+\varepsilon)^j d$. Since $d(l_i, s_\lambda) \leq l$, then $(1+\varepsilon)^\lambda d \leq l$ and

$$\lambda \leq \log_{1+\varepsilon}(l/d) = \frac{\ln(l/d)}{\ln(1+\varepsilon)} = O(\ln(l/d)/\varepsilon).$$

Hence each of the segments $\overline{l_i x}$ and $\overline{l_i y}$ contains $O(\ln(l/d)/\varepsilon)$ SPs. The segment $\overline{xy}$ contains $\lceil d(x, y)/(\varepsilon l) \rceil$ SPs, which number is $O(1/\varepsilon)$, since $d(x, y) < 2l$. □

5.2 Description and Analysis of the Algorithm

Next we define a weighted graph $G_\varepsilon = (V_\varepsilon, E_\varepsilon)$ called *approximation graph* with vertex set $\mathrm{Vor}(L) \cup \mathcal{S} \cup \{x, y\}$ and an edge between any pair of vertices corresponding to either points on different edges of the same triangle of $\mathcal{T}$ or to the same new edge. Add also edges joining vertices x and y to the SPs from the triangles containing x and y, respectively. Assign a weight $wt(v, w)$ on each edge (v, w) corresponding to the exposure along the minimum exposure path between v and w in the triangle containing v and w, if v and w belong to different edges, or along the segment containing v and w, otherwise.

Let Q denote the area of the region P and let q denote the minimum distance between any two points of L.

Lemma 6. *G_ε has $O(n/\varepsilon \ln(Q/q))$ vertices and $O(n/\varepsilon^2 \ln^2(Q/q))$ edges.*

Proof. By Lemma 5 each triangle contains $O(\ln(l/d)/\varepsilon) = O(\ln(\sqrt{Q}/q)/\varepsilon)$ SPs, as P has a bounded aspect ratio. Since $|\mathrm{Vor}(L)| = O(n)$, then the total number of triangles is $O(n)$. The lemma follows. □

Given G_ε, we compute an approximate minimum exposure path between x and y as a shortest path p_ε in G_ε between x and y. That shortest path can be computed using Dijkstra's shortest path algorithm [2] in $O(m_\varepsilon + n_\varepsilon \log n_\varepsilon)$ time, where $n_\varepsilon = |V(G_\varepsilon)|$ and $m_\varepsilon = |E(G_\varepsilon)|$.

Next we analyze how closely p_ε approximates the MEP. For each path p in G_ε let $wt(p)$ denote the sum of the weights of the edges of p.

Lemma 7. *For any $\varepsilon > 0$ there exists a path p in G_ε between vertices x and y such that $wt(p) \le (1 + O(\varepsilon/\breve{\alpha})) \exp(\mathrm{MEP}(x, y))$, where $\breve{\alpha}$ is the minimum angle of $\mathcal{T}$.*

Proof. Assume x and y belong to the interior of different triangles of $\mathcal{T}$ and let Δ_x and Δ_y be the triangles containing x and y, respectively. According to Lemma 2, $\mathrm{MEP}(x, y)$ consists of a sequence of subpaths $p_1, \ldots, p_\lambda$ in P, where p_1 and p_λ are MEPs between x or y and a point from Δ_x or Δ_y, respectively, and each of the other paths is either a MEP between points belonging to different edges of a triangle, or is a subsegment of an edge of $\mathcal{T}$. We will construct a sequence $p'_1, \ldots, p'_\lambda$ of paths in G_ε whose concatenation results in a path from x to y and such that $wt(p'_i) \le (1 + O(\varepsilon/\breve{\alpha}))\exp(p_i)$ for any $1 \le i \le \lambda$, which will imply to validity of the lemma. (If x or y is on an edge of the triangulation, then the corresponding paths p'_1 or $p'_{\lambda'}$ will be null paths.)

Choose any $i \in [1, \lambda]$ and consider the case where p_i connects points belonging to different new edges (l, z_1) and (l, z_2) of a triangle Δ, where $l \in L$ (Figure 2). (The proofs for the other cases are similar.) Let h_1 and h_2 be the source and the target of p_i and let h'_1 and h'_2 be the Steiner points on the segments $\overline{h_1 z_1}$ and $\overline{h_2 z_2}$ that are closest to h_1 and h_2, respectively. Denote $d(l_i, h_j) = \eta_j$ and $d(l_i, h'_j) = \eta'_j$ for $j = 1, 2$. By (11) $\eta'_j \le (1 + \varepsilon)\eta_j$ for $j = 1, 2$. Define a polar coordinate system with origin l and polar axis $\overline{l z_1}$. Let $\alpha = \angle z_1 l z_2$. Let $p_i = \{(\rho(\theta), \theta) \mid \theta \in [0, \alpha]\}$.

We will define a path $\bar{p}_i = \{(\overline{\rho}(\theta), \theta) \mid \theta \in [0, \alpha]\}$ with source $(h'_1, 0)$ and target (h'_2, α) that will be a "scaled up" version of p_i. More precisely, our goal is to define a function $k(\theta)$ such that the path defined by the function $\overline{\rho}(\theta) = k(\theta)\rho(\theta)$ will have exposure at most $1 + O(\varepsilon/\check{\alpha})$ times the exposure of p_i. We will show that it suffices that the following conditions are satisfied:

(i) $\overline{\rho}(0) = \eta'_1$ and $\overline{\rho}(\alpha) = \eta'_2$;

(ii) $\overline{\rho}^{\,-\mu} = (1 + O(\varepsilon))\rho^{-\mu}$;

(iii) $\sqrt{1 + \left(\frac{\overline{\rho}'}{\overline{\rho}}\right)^2} \leq (1 + O(\varepsilon/\check{\alpha}))\sqrt{1 + \left(\frac{\rho'}{\rho}\right)^2}$

We will prove that the function $k(\theta) = \frac{\alpha-\theta}{\alpha}\frac{\eta'_1}{\eta_1} + \frac{\theta}{\alpha}\frac{\eta'_2}{\eta_2}$ satisfies those conditions. A direct substitution shows that condition (i) is satisfied. Furthermore,

$$k(\theta) \leq \frac{\alpha - \theta}{\alpha}(1 + \varepsilon) + \frac{\theta}{\alpha}(1 + \varepsilon) = 1 + \varepsilon$$

and hence $\overline{\rho}/\rho \leq 1 + \varepsilon$ and (ii) holds. Property (iii) follows from the previous inequality $k(\theta) \leq 1 + \varepsilon$ and from

$$k'(\theta) = \left(\frac{\eta'_2}{\eta_2} - \frac{\eta'_1}{\eta_1}\right)\frac{1}{\alpha} = O(\varepsilon/\check{\alpha}).$$

By Properties (ii) and (iii)

$$\exp(\bar{p}_i) = \int_0^\alpha \overline{\rho}^{\,-\mu}\sqrt{1 + \left(\frac{\overline{\rho}'}{\overline{\rho}}\right)^2}\mathrm{d}\theta = \int_0^\alpha (1 + O(\varepsilon/\check{\alpha}))\rho^{-\mu}\sqrt{1 + \left(\frac{\rho'}{\rho}\right)^2}\mathrm{d}\theta$$
$$= (1 + O(\varepsilon/\check{\alpha}))exp(p_i).$$

Since by (i) p'_i and $\bar{p}_i$ are paths with the same source and target h'_1 and h'_2,

$$wt(p'_i) = \exp(MEP(h'_1, h'_2)) \leq \exp(\bar{p}_i) \leq (1 + O(\varepsilon/\check{\alpha}))\exp(p_i). \tag{12}$$

To complete the proof, we add together the inequalities (12) for $i = 1, \ldots, \lambda$. □

Combining Lemma 7 with our analysis of the time complexity, we get the following theorem.

Theorem 1. *Given a convex polygon P of bounded aspect ratio, a sensor field $F = (P, L, S)$ with nonnegative sensibility exponent, two points x and y from P, and any $\varepsilon > 0$, a path p in P between x and y such that* $\exp(p) \leq (1 + \varepsilon)\exp(p_{opt})$ *can be found in $O(n/\varepsilon^2 \ln^2(\psi)\check{\alpha}^2)$ time, where* $p_{opt} = \mathrm{MEP}(x, y, F)$, *$n = \max\{|L|, |P|\}$, ψ denotes the ratio of the area of P and the minimum distance between any two points of L, and $\check{\alpha}$ is the minimum angle of the triangulation of* $\mathrm{Vor}(L)$.

Note that although the justification of Theorem 1 is relatively complex, the implementation of the corresponding algorithm requires only running a shortest path algorithm on G_ε.

5.3 Improving the Running Time

The graph G_ε has a relatively high number of edges compared to the number of its vertices. This is due to the fact that in each triangle of $\mathcal{T}$ the number of the edges is roughly proportional to the square of the number of the Steiner points. On the other hand, the set of all shortest paths in $G\varepsilon$ has a structure that allows an efficient implementations of Dijkstra's shortest path algorithm that considers only a fraction of the edges of G_ε.

We will describe the idea of the algorithm BUSHWHACK [13] designed for solving shortest path problems for Euclidean-like distances and show how it can be modified in order to work in our case. The goal is to reduce the number of the edges considered to be roughly proportional to the number of the SPs (within a logarithmic factor). We will only consider here the case $\mu = 0$, where μ is the sensibility exponent from (1). The algorithm for arbitrary values of μ is similar, but needs a more complicated analysis because of the lack of simple analogue of the exposure formula (6).

The BUSHWHACK algorithm is based on Dijkstra's algorithm, which divides the vertices of the graph into two subsets: U, containing vertices to which the exact distances $d_{G_\varepsilon}(x, s)$ from the source x have already been computed, and $V \setminus U$, containing vertices to which approximate distances from x have been assigned based on paths restricted to contain vertices from U only. At each iteration a vertex $s \in V \setminus U$ with a minimum current distance from x is moved to U and the distances to the neighbors of s in $V \setminus U$ are updated.

In order to introduce the BUSHWHACK modification to Dijkstra's algorithm, consider any triangle $\Delta \in \mathcal{T}$. If e is an edge of Δ, we denote by $V(e)$ the set of the vertices of G_ε that correspond to SPs from e. For any two edges e and e' from Δ and vertex v on e that is not on e' we define the set $I(v, e, e')$ consisting of all vertices z from e' such that

$$d_{G_\varepsilon}(x, v) + wt(v, z) \leq d_{G_\varepsilon}(x, v') + wt(v', z)$$

for any vertex v' from e. The sets $I(v, e, e')$ can be used to reduce the number of the edges in Δ considered by Dijkstra's algorithm, since for any vertex z from $I(v, e, e')$ there is a shortest paths from x to z that does not contain any vertex from $V(e) \setminus \{v\}$. Hence all edges connecting a vertex from $V(e) \setminus \{v\}$ to a vertex from $I(v, e, e')$ can be ignored in the shortest paths computation.

In fact, the sets $I(v, e, e')$ are dynamic and are updated each time when a new vertex from $V(e)$ is added to U. In order to ensure that these sets can be maintained efficiently, we need to prove that the following two properties hold. Let $\pi(v, w)$ denote the path in P resulting from replacing each edge of the shortest path in G_ε between v and w with the corresponding minimum exposure path and let $d'(v, w)$ denote the exposure of $\pi(v, w)$ (which is also equal to the distance between v and w in G_ε).

Lemma 8. *Let $\pi_1 = \pi(x, y_1)$ and $\pi_2 = \pi(x, y_2)$. Let, for $i = 1, 2$, π_i intersects the edges of a triangle Δ of $\mathcal{T}$ at vertices z_{i1} and z_{i2}, respectively (Figure 1), where all vertices z_{ij}, $1 \leq i, j \leq 2$, are distinct. Then the segments $\overline{z_{11}z_{12}}$ and $\overline{z_{21}z_{22}}$ do not intersect.*

Lemma 8 implies that each set $I(v,e,e')$ consists of consecutive points on e', i.e., no vertex of $V(e) \setminus I(v,e,e')$ is between two vertices from $I(v,e,e')$ on e'. Hence $I(v,e,e')$ can be identified with an interval (e.g., a pair of points) on e'. The next lemma can be used to compute and maintain such intervals efficiently (in $O(\log|V(e')|)$ time).

A segment s is called *monotonic* [13] with respect to a point z, if the exposure from z to points of s is either monotonically increasing or monotonically decreasing along s.

Lemma 9. *If s is a segment belonging to a line containing the sensor location l_i, then s can be divided into two monotonic segments with respect to any point z in $O(1)$ time.*

Proof. The point z' such that $d(O,z') = d(O,z)$ divides s into monotonic segments, if z is not on s. If z is on s, then s itself is monotonic. ◻

The proof of an analogue of Lemma 9 for the case where the line containing s does not contain l_i is more complex. Instead of proving such lemma, we notice that we can use other properties to define the set $I(v,e,e')$ in the case where e' is a segment of $V(L)$ and v does not belong to e'. We consider the following two cases for v.

(i) $v = x$. In this case we define $I(v,e,e') = e'$ since $U = \{v\}$.
(ii) $v \neq x$. Note that v can not be either of the endpoints of e as by assumption v is not on e' and v can not be a sensor location l_i as by construction $L \cap \mathcal{S} = \emptyset$ (see (11)). Then v is an internal point of e. Denote by v^* the predecessor of v on the shortest path from x to v. That vertex must have already been determined by the algorithm since $v \in U$. Then v is the closest SP to the intersection point between e and a MEP determined by formulas (4)–(5) from v^* to a point on e'. Then $I(v,e,e')$ can be determined as the smallest segment on e' whose endpoints are SPs and which contains the intersection point of e' and the MEP determined by v and v^*.

Further details on the data structures and the analysis of BUSHWHACK algorithm can be found in [13]. We established the following result, which is an improvement of Theorem 1 for the case $\mu = 0$ by a factor of roughly $\Theta(1/\varepsilon)$.

Theorem 2. *Given a polygon P of bounded aspect ratio, a sensor field $F = (P, L, S)$ with zero sensibility exponent, two points x and y from P, and any $\varepsilon > 0$, a path p between x and y in P such that $\exp(p) \leq (1+\varepsilon)\exp(p_{opt})$ can be found in $O(m \log m)$ time, where $p_{opt} = \mathrm{MEP}(x,y,F)$, $n = \max\{|L|,|P|\}$, $m = O(n/\varepsilon \ln(\psi)\breve{\alpha})$, ψ denotes the ratio of the area of P and the minimum distance between any two points of L, and $\breve{\alpha}$ is the minimum angle of the triangulation of $\mathrm{Vor}(L)$.*

6 Conclusion

In this paper we developed the theoretic framework for designing approximation algorithms for solving minimum exposure path problems for sensor networks.

There are several interesting problems not discussed here that will be subject of our ongoing and future work. These include removing the dependence of the running time of Theorem 2 on $\breve{\alpha}$. Although such a dependence is characteristic for such type of problems, e.g., [3], we will show in the full version of this paper that it can be removed in our case. We also plan to use our MEP algorithms for solving placement problems for sensor networks.

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