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Planar Crossing Numbers of Graphs of Bounded Genus

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Abstract Pach and Tóth proved that any n-vertex graph of genus g and maximum degree d has a planar crossing number at most $c^g dn$, for a constant $c > 1$. We improve on this results by decreasing the bound to $O(dgn)$, and also prove that our result is tight within a constant factor. Our proof is constructive and yields an algorithm with time complexity $O(dgn)$. As a consequence of our main result, we show a relation between the planar crossing number and the surface crossing number.

Keywords Crossing number · Genus · Orientable surface · Surface crossing number

Mathematics Subject Classification (2000) 05C10 · 68R10 · 05C85

1 Introduction

A *drawing* of a graph G in the plane is an injection of the set of the vertices of G into points of the plane and a mapping of the set of the edges of G into simple continuous curves such that the endpoints of each edge are mapped onto

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the endpoints of its image curve. Moreover, no curve should contain an image of a vertex in its inside and no three curves should intersect in the same point, unless it is an endpoint. The *planar crossing number* (or simply the *crossing number*) of G, denoted by $\mathrm{cr}(G)$, is the minimum number of edge crossings over all drawings of G in the plane.

The concept of crossing numbers was introduced [26] more than 50 years ago by Turán. Although there have been scores of results and publications since, because of the difficulty of the problem there are only a few infinite classes of graphs with determined exact crossing numbers. For instance, Glebsky and Salazar recently proved that the crossing number of the Cartesian product of two cycles $C_m \times C_n$ is $(m-2)n$ [11]. But the exact crossing numbers for such important graphs as the complete graph K_m and the bipartite graph $K_{m,n}$ are not known, in general. For an annotated bibliography of crossing number results see [16] and for a more extensive and up-to-date chronological bibliography see [23]. A recent survey paper on crossing numbers is in [25].

From algorithmic point of view, crossing numbers have been studied by Leighton [15], who was motivated by their application in VLSI design. In graph drawing, crossing numbers have been used for finding aesthetic drawing of nonplanar graphs and graph-like structures [6]. Typically, such graphs are drawn in the plane with a small number of crossings and, next, each crossing point is replaced by a new vertex of degree 4. The resulting planar graph is then drawn in the plane using an existing algorithm for planar graph drawing. Finally, the new vertices are removed and replaced back by edge crossings. The general drawing heuristics are usually based on the divide and conquer approach, using graph separators, or using 2-page layouts [5,15].

The problem of finding the crossing number of a given graph was first proved to be NP-hard by Garey and Johnson [10] and, more recently, it was shown to be NP-hard even for cubic graphs [13]. There is only one exact algorithm of practical use [3], but it works for small and sparse graphs only. The best polynomial algorithm approximates the crossing number with a polylogarithmic factor [9,4]. A quadratic, fixed parameter-tractable algorithm for crossing numbers was found in [12]. Kawarabayashi and Reed [14] construct for every fixed k a linear time algorithm that constructs a drawing of an input graph in the plane with at most k crossings or determines that such a drawing does not exist, answering a question posed in [12].

Another direction of research is to estimate crossing numbers in terms of basic graph parameters, like density, separators, cutwidths and edge congestions. There are only a few results of this type [1,15,20,7,17]. And although the crossing number and the genus of the graph are two of the most important measures for nonplanarity, there are only a few results that study the relationship between them. Pach and Tóth [19] showed that any n-vertex *toroidal* graph G (i.e., graph that can be drawn on the torus with no intersections) of maximum degree d has crossing number $O(dn)$. If G is of orientable genus g (i.e., can be drawn on an orientable surface S_g of genus g with no intersections), they proved that $\mathrm{cr}(G) \leq c^g dn$, for some constant $c > 1$. Unfortunately, the constant c is very large and, as a consequence, their result can be useful for very small values of g only. Although their proofs are of a constructive type, Pach and Tóth do not discuss algorithmic issues. Later on, they proved a similar result for nonorientable surface too, [2]. Another result of this nature estimates planar crossing numbers of H-minor-free graphs [24].

In this paper we show that

$$\mathrm{cr}(G) = O(dgn)$$

and that the bound is tight within a constant factor. The proof is of algorithmic nature. Our result is also interesting because of the fact that it relates the crossing numbers of a given graph on two different surfaces. Specifically, let $\mathrm{cr}_g(G)$ denote the orientable surface g crossing number of G, i.e., the minimum number of edge crossings over all drawings of G in S_g. The above type of results says that if $\mathrm{cr}_g(G) = 0$, then $\mathrm{cr}(G)$ cannot be very large. We further strengthen this result by showing as a corollary of our main result that $\mathrm{cr}(G) = O(\mathrm{cr}_g(G)\, g + gn)$ for bounded degree graphs.

This paper is organized as follows. In Section 2 we give some basic definitions and facts about embeddings and surfaces. In Section 3, we prove our main result and describe the drawing algorithm based on our upper bound proof. In Section 4, we give a lower bound proof that shows the tightness of our bound. We conclude with a discussion of extensions and generalizations of the results presented in the paper.

2 Preliminaries

A *graph* is an ordered pair of sets V and E, where V is the set of the *vertices* of the graph and E is the set of the *edges*. Each edge e is a pair of vertices v and w. If the pair is ordered, the edge, denoted by $e = \langle v, w\rangle$, is *directed*, called also an *arc*, and if the pair is unordered, the edge, denoted by $e = (v, w)$, is *undirected*. The graph is *undirected* if all its edges are undirected, and otherwise the graph is *directed*.

A *path* P is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $i < k$. If $v_0 = v_k$, then P is a *cycle* and P is a *simple* cycle if all vertices $v_1, \ldots, v_k$ are distinct.

The maximal connected subgraphs of G are its *components*. A connected graph is *biconnected*, if the removal of any vertex leaves a connected graph. The maximal biconnected subgraphs of G are its *bicomponents*.

In this paper, unless stated otherwise, by G we denote an undirected connected graph and by $V(G)$ and $E(G)$ we denote the set of the vertices and the set of the edges of G, respectively. The *size* of G is the number of its edges. For any vertex v, the number of the adjacent vertices to v is called the *degree* of v and is denoted by $\deg(v)$. The maximum degree of any vertex of G is called the *degree* of G. The set of the vertices adjacent to v is called the *neighborhood* of v and is denoted by $N(v)$. For any set of vertices X, the *neighborhood* of X is $N(X) = \bigcup_{v\in X} N(v)$.

The *bisection width* of G, denoted by $\mathrm{bw}(G)$, is the smallest number of edges whose removal divides the graph into parts having no more than $2|V(G)|/3$ vertices each.

By a *surface* we mean a closed manifold and by S_g we denote an orientable surface of genus g. A *drawing* of G on S_g is any injection of the vertices of G onto points of S_g and the edges of G onto continuous simple curves of S_g so that the endpoints of any edge are mapped onto the endpoints of its corresponding curve. The drawing is called an *embedding*, if no two curves intersect, except possibly at an endpoint. An embedding is called a *2-cell embedding* if every component of $S_g \setminus G$ is homeomorphic to an open disk. The *genus* of G, denoted by $\gamma(G)$, is the smallest genus of a surface G can be embedded in. G is *planar*, if the genus of G is zero. Every planar graph can be drawn in the plane without any edge intersections.

Throughout this paper we will use a *combinatorial representation* (also called a *rotation system*) of a 2-cell embedding, which describes the circular ordering of the edges incident to each vertex. Specifically, let G be a graph and let $\mu(G)$ be a 2-cell embedding of G. In order to construct the combinatorial representation of $\mu(G)$, replace each undirected edge (v, w) of G by a pair of opposite arcs $\langle v, w\rangle$ and $\langle w, v\rangle$, and denote the set of the resulting arcs by $\tilde{E}$. The combinatorial representation of $\mu(G)$ (denoted simply by $\mu(G)$ hereafter) is defined by the set of the cyclic lists of arcs of $\tilde{E}$, called *arc orbits*, where each orbit lists the outgoing arcs from any vertex v, in the order in which they appear around v in the embedding in a counterclockwise direction. If e' is the first edge in the edge orbit of v from e, then we will write $e \xrightarrow{v} e'$. A *facial walk* of $\mu(G)$ is any sequence of arcs from $\tilde{E}$, where the successor of any arc $\langle v, w\rangle$ is the arc after $\langle w, v\rangle$ in the arc orbit for w. The *faces* of embedding are all simple closed facial walks and they correspond to the maximal connected regions into which the drawing of G divides the surface. Note that the set of all facial walks contain each edge exactly twice. The *outer face* of a planar embedding corresponds to the infinite face of the corresponding drawing. In a combinatorial embedding, any face can be chosen to be the outer face.

In the remainder of this paper, we will use n, m, g, and d to denote the numbers of vertices, edges, the genus, and the degree of G, respectively. We also assume that we are given an embedding $\mu(G)$ of G in S_g as an input. If f denotes the number of the faces of $\mu(G)$, then the *Euler characteristic* $\mathcal{E}(\mu(G))$ of $\mu(G)$, denoted simply by $\mathcal{E}(G)$ when the embedding is clear from the context, is defined as

$$\mathcal{E}(G) = \mathcal{E}(\mu(G)) = n - m + f.$$

The relation between the Euler characteristic and the genus g of the embedding of a graph of k components is given by the *Euler formula*

$$n - m + f = 2k - 2g. \tag{1}$$

If G' is a subgraph of G, the embedding $\mu(G')$ of G' *induced* by $\mu(G)$ is defined by the modified orbits of $\mu(G)$, where in each orbit the edges in G' are kept and the ones not in G' are skipped. The genus of the induced embedding can be determined by computing the number of its faces and applying the Euler formula.

For any subgraph K of G, let $\mu(K)$ denote the embedding of K induced by $\mu(G)$, let $\gamma_\mu(K)$ denote the genus of $\mu(K)$, and let $\gamma(K)$ denote the genus of K. Note that $\gamma_\mu(K)$ and $\gamma(K)$ may not be equal. In order to simplify notations, we may denote $\gamma_\mu(K)$ simply by g_K, if $\mu(K)$ is clear from the context.

3 The drawing algorithm

This is the main section of the paper, where we describe and analyze an algorithm for drawing into the plane with a small number of crossings a graph G embedded in S_g. We will start in Section 3.1 by describing a procedure for partitioning G into components with special properties, which we divide into three classes of components. In Section 3.2 we will outline the rest of the algorithm that draws each component according to its type and then combines all drawings into a drawing of the original graph.

3.1 Cutting the graph into components and analyzing their properties

Without a loss of generality, we assume that G is biconnected, since otherwise one can draw the biconnected components separately in the plane and then combine their drawings into a planar drawing of G with no additional crossings. Triangulate $\mu(G)$ by inserting a suitable number of additional edges in each face that is not a triangle. Assign weights 1 to all original edges of G and weights 0 to all new edges.

In order to simplify the notations, we will continue to denote by G and $\mu(G)$ the modified graph and embedding, respectively, and will refer to the edges of weights 1 and 0 as *original* and *new* edges of G, respectively. For any set $X \subseteq E(G)$, let $wt(X)$ denote the sum of the weights of all edges of X. Since in our algorithms we will only be interested in crossings between original edges of G, we introduce the term *original crossings* to refer to crossings where both intersecting edges are original.

Select any vertex t and divide the vertices of G into *levels* according to their distance to t. For an integer constant r to be determined later, denote by L_j, for $0 \leq j < r$, the set of all edges between level i and level $i+1$ vertices, for all integers i satisfying $i \bmod r = j$. Assume that the number of all levels is at least r. Then there exists an $i^* < r$ such that

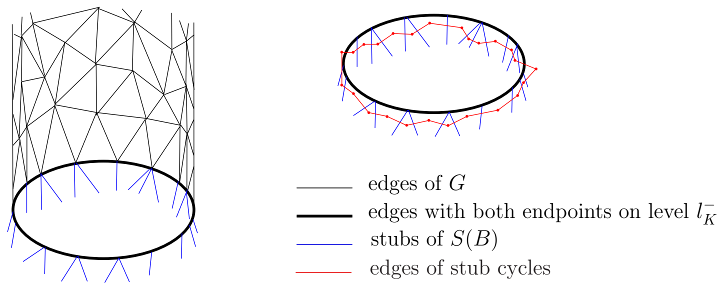
$$wt(L_{i^*}) \leq \lfloor m/r \rfloor. \tag{2}$$

Replace each edge $e = (v, w) \in L_{i^*}$ by a pair of new edges $s_1 = (v, x_1)$ and $s_2 = (w, x_2)$ called *stubs*, where x_1 and x_2 are new vertices. This has the effect of "cutting" e. The stubs s_1 and s_2 are called a *matching pair* of stubs and e is called a *parent* of s_1 and s_2. For any stub (v, x), where $v \in V(G)$ and $x \notin V(G)$, vertex v is called *attached* and vertex x is called *unattached*. Our drawing algorithm will eventually join each pair of stubs back into their parent edge.

Next we are going to analyze some relevant properties of the resulting graph, which we denote by G'. Compute the set $K(G')$ of all connected components of G'. Let $K \in K(G')$. Denote by l_K^- and l_K^+ the lowest level and the highest level of the vertices of K, respectively. Let $\mu(K)$ be the embedding of K induced by the embedding of G, where stubs take the places of their parent edges in the edge orbits of the vertices on levels l_K^- and l_K^+. Denote by $\mathcal{B}(K)^-$ (respectively $\mathcal{B}(K)^+$) the set of all bicomponents of the subgraph of G induced by the vertices on the lowest (respectively highest) level of K that are incident to stubs.

Let B be any bicomponent of $\mathcal{B}(K)^-$. Let $S(B)$ be the set of all stubs adjacent to a vertex of B and let $S(B)'$ be the set of the parent edges of all stubs from $S(B)$. Assign level l_K^- to any new vertex that is an endpoint of a stub from $S(B)$. We will define a decomposition $\mathcal{C}(B)$ of $S(B)$ into subsets as follows. Intuitively, the elements of $\mathcal{C}(B)$ will correspond to new faces determined by $S(B)$ and the embedding of G (Figure 1 (a)). Define the dual graph of $\mu(G)$ and construct its subgraph, which we call G_B, containing all triangles whose vertices are on levels in $\{l_K^- - 1, l_K^-\}$ and have at least one vertex from B and at least one vertex on level $l_K^- - 1$ (Figure 1 (b)). Clearly, each edge of $S(B)'$ belongs to a face dual to a vertex in G_B. The next lemma further characterizes the relationship between $S(B)'$ (or $S(B)$) and G_B.

Lemma 1 *Each vertex of G_B is adjacent to exactly two edges dual to edges from $S(B)'$.*



(a) (b)

Fig. 1 The set $S(B)$ and its corresponding face of $\mu(K)$. (b) The stub cycle corresponding to $S(B)$. In general, $S(B)$ may define multiple stub cycles that are not necessarily simple.

Proof Let x be any vertex of G_B, let $f = (u, v, w)$ be the face dual to x. Denote by n_i and n_{i-1} the number of the vertices in the set $\{u, v, w\}$ on levels i and $i-1$, respectively, where $i = l_K^-$. By the definition of G_B, either $n_i = 2$ and $n_{i-1} = 1$ or $n_i = 1$ and $n_{i-1} = 2$. From the definition of $S(B)'$, exactly two edges in $\{(u, v), (v, w), (u, w)\}$ are in $S(B)'$. The lemma follows. ◻

The claim of Lemma 1 is also true if B is a bicomponent of $\mathcal{B}(K)^+$. In the latter case, we define $S(B)$ to be the set of all edges joining a vertex of B to a vertex on a level $l_K^+ + 1$. Let, for any B from $\mathcal{B}(K)^-$ or $\mathcal{B}(K)^+$, G'_B denote the subgraph of G_B induced by the set of the edges $S(B)$.

Corollary 1 *G'_B is a union of edge disjoint cycles and has the same vertices as G_B.*

Let $\mathcal{C}(B)$ denote the set of the cycles comprising G'_B, which we call *stub cycles* (Figure 1 (b)). The next lemma analyzes the changes in G that will take place if only those edges of G corresponding to a single stub cycle were replaced by stubs.

Lemma 2 *Let C be a stub cycle from $\mathcal{C}(B)$, let $E(C)$ be the edges of C, let $E(C)'$ be the set of edges in $S(B)$ dual to $E(C)$, and let $E(C)''$ be the parent edges of the edges in $E(C)'$. Replace in G all edges from $E(C)''$ with their corresponding stubs. Let G_c be the resulting graph and k_c and g_c be the number of components and the genus of G_c, respectively. Then $g_c - k_c = g - 2$.*

Proof Let $s = |E(C)''|$, let $i = l_K^-$, and let n_c, m_c, and f_c denote the number of the vertices, edges, and faces of $\mu(G_c)$. By construction, any edge from $E(C)''$ is split into two new edges (stubs), thereby increasing the number of the vertices by two and the number of the edges by one. Hence

$$n_c = n + 2s$$

$$m_c = m + s.$$

Let S_1 denote the set of the stubs corresponding to the edges of $E(C)''$ that are incident to vertices on level i and let S_2 denote the corresponding stubs that are incident to vertices on level $i-1$. By Lemma 1, each face of $\mu(G)$ that is incident to a stub from S_1 is incident with exactly two stubs from S_1. Furthermore, since by construction $\mu(G)$ is a triangulation, each stub of S_1 is incident with exactly two faces of $\mu(G)$. To prove that, assume that there is a stub (x, y) such that both arcs $\langle x, y\rangle$ and $\langle y, x\rangle$ appear in the same face, say f. Then the facial walk corresponding to f should be either $(\langle x, y\rangle\langle y, y\rangle\langle y, x\rangle)$ or $(\langle x, y\rangle\langle y, x\rangle\langle x, x\rangle)$, both of which are impossible as G has no self-loops.

Hence, the number of faces in $\mu(G)$ incident to stubs from S_1 is exactly equal to the number of the stubs of S_1, which is s. Furthermore, these s faces of $\mu(G)$ are replaced by two new faces of $\mu(G_c)$ that are not in $\mu(G)$, namely, one containing all stubs from S_1 and the other containing all stubs from S_2. Hence

$$f_c = f - s + 2\,.$$

Combining the last three equalities we get

$$n_c - m_c + f_c = n - m + f + 2\,.$$

Finally, applying the Euler formula on the left-hand and right-hand sides of the last equation, we get

$$2k_c - 2g_c = 2 - 2g + 2\,,$$

implying the claim of the lemma. □

For any component K of G', let c^-_K denote the sum of $|\mathcal{C}(B)|$ for all bicomponents $B \in \mathcal{B}(K)^-$. Let $c^-_{G'}$ denote the sum of c^-_K over all components K of $K(G')$. We can think of $c^-_{G'}$ as the number of the handles cut by the division of the graph (the replacement of edges by stubs). Denote $g' = \gamma_\mu(G')$, where $\gamma_\mu(G')$ by definition denotes the genus of the embedding $\mu(G')$ induced by $\mu(G)$. Then the following equality gives the relationship between the genus before and after the division, the number of handles cut, and the number of the resulting components.

Lemma 3 $g' = g - c^-_{G'} + |K(G')| - 1$.

Proof Let K be any component of $K(G')$ that does not contain the vertex on level zero (i.e., such that $c^-_K > 0$), let $B \in \mathcal{B}(K)^-$, and let $C \in \mathcal{C}(B)$. Construct the graph G_c and let g_c and k_c be as defined in Lemma 2. By Lemma 2

$$g_c - k_c = g - 2\,.$$

Applying this operation for all $C \in \mathcal{C}(B)$, $B \in \mathcal{B}(K)$, and $K \in K(G')$, we obtain a $|K(G')|$-component graph of genus $\gamma_\mu(G')$. By induction from the last equality,

$$g' - |K(G')| = g - 2 - (c^-_{G'} - 1) = g - c^-_{G'} - 1,$$

implying the correctness of the claim. □

The next corollary establishes the intuitively evident fact that replacing the edges of L_{i^*} by stubs doesn't increase the genus of the resulting induced embedding.

Corollary 2 $g' \leq g$.

Proof Follows from Lemma 3, since $|\mathcal{B}(K)^-| \geq 1$ for each component $K \in K(G')$ except the one containing vertex t and hence $c^-_{G'} \geq |K(G')| - 1$. □

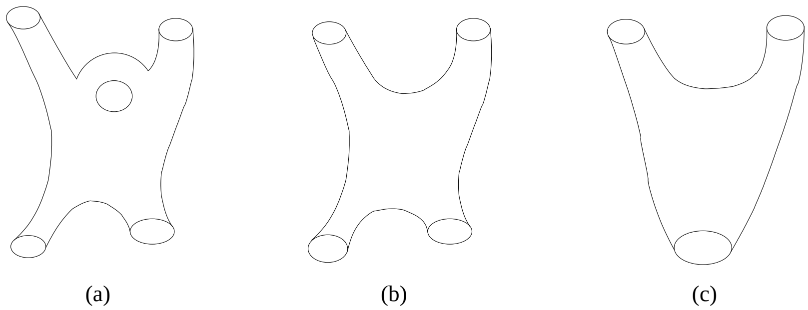


Fig. 2 The three types of components with respect to the genus of their embeddings and the number of stub cycles: (a) non-planar component; (b) m-planar component; (c) s-planar component.

Next we are going to estimate the genus of $\mu(K)$. For any component K of G', let n_K and m_K be the numbers of the vertices and the edges of K, respectively, and let f_K be the number of the faces of $\mu(G)$ whose all edges are in K.

Lemma 4 *The Euler characteristic of* $\mu(K)$ *is* $\mathcal{E}(K) = n_K - m_K + f_K + c_K^- + c_K^+$.

Proof As each element of $\mathcal{B}(K)$ is a cycle that defines a single new face of $\mu(K)$, $\mu(K)$ has $c_K^- + c_K^+$ faces that are not faces of $\mu(G)$. By (1), the Euler characteristic of $\mu(K)$ is $\mathcal{E}(K) = n_K - m_K + (f_K + c_K^- + c_K^+)$. The claim follows. □

Corollary 3 *The genus of* $\mu(K)$ *is* $g_K = 1 - (n_K - m_K + f_K + c_K^- + c_K^+)/2$.

Proof Follows from Lemma 4 and the Euler formula. □

We will divide the set $K(G')$ of all components K of G' into three types depending on c_K^- and on g_K as follows (Figure 2).

(i) If $g_K > 0$, then K will be of *non-planar* type (note that K can actually be planar if the genus of K is smaller than the genus of the induced embedding $\mu(K)$). Let C_{np} denote the set of all non-planar components K of G'.
(ii) If $g_K = 0$ and $c_K^- > 1$, then K will be of *m-planar* type (for" multi-cycle planar," referring to the set of stubs adjacent to the lowest level). Let C_{mp} denote the set of all m-planar components.
(iii) If $g_K = 0$ and $c_K^- = 1$, then K will be of *s-planar* (for "single-cycle planar") type. Let C_{sp} denote the set of all s-planar components.

Note that it will be more accurate to say that $\mu(K)$ is of the given type instead of K, because the type depends on the embedding rather than the graph itself. However we use the above terminology to simplify the notations and use a single embedding μ throughout the paper.

Next we are going to estimate the sizes of C_{np} and C_{mp} and the numbers of stub cycles for components of those types. First we will look at C_{np}. It is natural to expect that there cannot be more than g non-planar components of G'. To prove that formally, we will make use of the Euler formula.

Lemma 5 $|C_{np}| \leq g$.

Proof Denote by n, m, f and by n', m', f' the number of the vertices, edges, and faces of $\mu(G)$ and $\mu(G')$, respectively. Denote $k' = |K(G')|$. By the Euler formula

$$n' - m' + f' = 2k' - 2g'. \tag{3}$$

Let $C_p = C_{mp} \cup C_{sp}$, i.e., the set of the "planar" components K of $\mu(G')$ for which $g_K = 0$. Denote by n_p, m_p, f_p and by n_{np}, m_{np}, f_{np} the number of the vertices, edges, and faces of in C_p and C_{np}, respectively. Then, by adding the Euler formulas for all components of K_p and K_{np}, we get, respectively,

$$\begin{aligned} n_p - m_p + f_p &= 2|C_p| = 2(k' - |C_{np}|) \\ n_{np} - m_{np} + f_{np} &\le 0\,, \end{aligned}$$

as the Euler characteristic of any nonplanar embedding is non-positive. Adding the last two formulas gives

$$n' - m' + f' \le 2k' - 2|C_{np}|.$$

By combining the last inequality with (3) we get

$$2k' - 2g' \le 2k' - 2|C_{np}|$$

and, by Corollary 2,

$$|C_{np}| \le g' \le g.$$

□

The next result strengthens the previous lemma by showing that not only the number of the components K in C_{np}, but also the number c_K^- of their stub cycles is $O(g)$.

Lemma 6 $\displaystyle\sum_{K \in C_{np}} c_K^- \le 2g.$

Proof By Lemma 3,

$$c_{G'}^- - |K(G')| + 1 = g - g' \le g. \tag{4}$$

By definition

$$\begin{aligned} c_{G'}^- = \sum_{K \in K(G')} c_K^- &= \sum_{K \in C_{np}} c_K^- + \sum_{K \in K(G') \setminus C_{np}} c_K^- \\ &\ge \sum_{K \in C_{np}} c_K^- + (|K(G')| - 1) - |C_{np}| \\ &\ge \sum_{K \in C_{np}} c_K^- + |K(G')| - g - 1\,, \end{aligned} \tag{5}$$

where Lemma 5 was used for the last inequality. The claim follows by substituting the inequality (5) for $c_{G'}^-$ in (4). □

Next we prove an analogue of Lemma 6 for C_{mp}.

Lemma 7 $\displaystyle\sum_{K \in C_{mp}} c_K^- \le 2g.$

Proof By the definition of $c^-_{G'}$

$$\begin{aligned} c^-_{G'} &= \sum_{K\in C_{mp}} c^-_K + \sum_{K\in K(G')\setminus C_{mp}} c^-_K \\ &= \sum_{K\in C_{mp}} c^-_K + (|K(G')|-1) - |C_{mp}|. \end{aligned} \tag{6}$$

Since $c^-_K \geq 2$ for $K \in C_{mp}$, then

$$\frac{1}{2}\sum_{K\in C_{mp}} c^-_K \geq |C_{mp}|. \tag{7}$$

Adding (6) and (7) together gives

$$c^-_{G'} \geq \frac{1}{2}\sum_{K\in C_{mp}} c^-_K + |K(G')| - 1.$$

Substituting $c^-_{G'}$ from the last inequality into (4) results into

$$\frac{1}{2}\sum_{K\in C_{mp}} c^-_K \leq g.$$

□

Note that there is no analogue of Lemmas 6 and 7 for C_{sp}. In fact, $|C_{sp}|$ can be as big as $\Omega(m/r)$, which will be $\Omega(\sqrt{dgm})$ for the choice $r = \lceil \sqrt{m/(gd)}\, \rceil$ we will make in Section 3.6, i.e., $|C_{sp}|$ is not necessarily $O(g)$.

3.2 Algorithm outline

The rest of the algorithm draws each component K of G' in the plane according to its type. The goal is to have the unattached endpoints of all stubs drawn in the outer face with a relatively small number of original crossings between edges of K. After all components are drawn in such a way, all pairs of matching stubs are joined into their parent edges. As all stubs are going to be in the outer face already, intersections might occur only between pairs of stubs. Since, by (2), the weight of all stubs is $O(m/r)$, this final step will increase the total number of original crossings by $O((m/r)^2)$.

3.3 Drawing components of non-planar type

If K is of non-planar type, then we will show that a subgraph of K of relatively small size can be found such that "cutting" the embedding of K along the edges of that subgraph and appropriately "pasting" a single face f along the cut produces a planar surface. Then we will draw the modified K in the plane with f as an outer face and redraw the edges that were destroyed by the cut to get a drawing of the original K. Since those edges will be drawn entirely in f, they will not intersect other edges of K. Finally, we will route all stubs to the outer face.

3.3.1 Finding a planarizing set for K

Consider a component K such that $g_K > 0$ and let l_K^- and l_K^+ denote the lowest and the highest levels of K. Define a spanning forest F_K of K with c_K^- trees as follows.

1. For each stub cycle C defined by the vertices of K on level l_K^- define a spanning tree T_c for C and add the edges of T_c to F_K.
2. For each vertex v on level greater than l_K^- choose arbitrarily a single vertex w on a lower level adjacent to v and add edge (v, w) to F_K.
3. For each stub s from K, make the attached endpoint of s parent of its unattached endpoint.

Clearly, F_K contains c_K^- trees, one for each stub cycle induced by level l_K^- in K. We will call F_K*-cycle* any simple cycle in K that has exactly one non-forest edge. Since K has no-more than r levels, any F_K-cycle will contain no more than $2(r-1)$ vertices of K, excluding the vertices on level l_K^-.

For any non-forest edge e of K incident to two different faces f_1 and f_2 of $\mu(K)$, remove e and merge f_1 and f_2 into a single face. Since this operation eliminates one edge and one face, the Euler characteristic does not change. Continue until no such edge e remains. Then any of the remaining non-forest edges should be incident only to a single face, say f. Therefore, f should be the only face of the resulting embedding, since any face must contains a non-forest edge (otherwise T will contain a cycle). Next, iteratively remove any edge that is incident to a degree-1 vertex as well as the degree-1 vertex itself. Since each removal reduces the number of the vertices and the number of the edges by one, this operation preserves the Euler characteristic of $\mu(K)$.

Denote by $Pl(K)$ the resulting graph. We will think of $Pl(K)$ as a "planarizing" graph since, as we will show in Step 4.2, it can be used to transform the embedding of K into a planar embedding. Denote by n_{Pl} and m_{Pl} the number of the vertices and the number of the edges of $Pl(K)$. By (1), we have

$$n_{Pl} - m_{Pl} + 1 = 2 - 2g_K, \tag{8}$$

and hence

$$m_{Pl} = (n_{Pl} - 1) + 2g_K,$$

which implies that the number of the remaining non-forest edges is $2g_K$. Therefore, $Pl(K)$ is a union of $2g_K$ F_K-cycles.

We proved the following.

Lemma 8 *The embedding $\mu(Pl(K))$ of $Pl(K)$ has a single face, genus g_K, and no more than $4g_K(r-1)$ vertices whose levels are in the interval $(l_K^-, l_K^+]$.*

We will use $Pl(K)$ in the next subsection to "planarize" $\mu(K)$.

3.3.2 Transforming $\mu(K)$ into a planar embedding

Next we transform $\mu(K)$ by modifying $Pl(K)$ so that it is transformed into a single new face f bounded by a simple cycle C. See the example on Figure 5. We will now describe more formally the transformation of the different types of elements

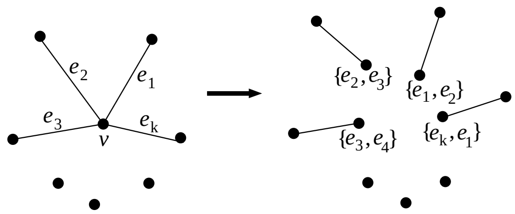


Fig. 3 Replacing vertex v from $Pl(K)$ by k new vertices.

of K. Vertices not in $Pl(K)$ and edges with no endpoint in $Pl(K)$ are not changed. The other vertices and edges of K are transformed as follows. Let E' be the set of the edges of K with at least one endpoint in $Pl(K)$.

(1) *Vertices of* $Pl(K)$. Let v be any vertex of degree k from $Pl(K)$ and let $< e_1, \cdots, e_k >$ be the counterclockwise permutation of the edges of $Pl(K)$ incident to v. Define k new vertices that will replace v and label them by $\{e_1, e_2\}, \{e_2, e_3\}, \cdots$, and $\{e_k, e_1\}$, respectively (Figure 3).

(2) *Edges from* $Pl(K)$. Let $e \in E(Pl(K))$ and let $e' \overset{w_2}{\to} e$ in $Pl(K)$, i.e., e is the first edge from $Pl(K)$ in a counterclockwise direction from e' in the edge-orbit of w_2. Let $e \overset{w_1}{\to} e''$ in $Pl(K)$ (Figure 4). Define a new edge $\overrightarrow{new}(w_2, w_1) = (\{e, e''\}, \{e', e\})$. Similarly, define an edge $\overrightarrow{new}(w_1, w_2)$ by swapping w_1 and w_2. Finally, replace e by the two edges $\overrightarrow{new}(w_2, w_1)$ and $\overrightarrow{new}(w_1, w_2)$ (note that both those new edges are undirected).

(3) *Edges in* $E' \setminus Pl(K)$. Let $e = (w_1, w_2) \in E' \setminus E(Pl(K))$. (Note that $e \notin Pl(K)$ does not preclude both endpoints of e to be in $Pl(K)$.) We will define an edge (w_1', w_2') to replace e, where w_1' and w_2' are determined as follows. If w_1 is not from $Pl(K)$, let $w_1' = w_1$. Else denote by $< e_1, \cdots, e_k = e_0 >$ the edge-orbit of w_1 and let $e \overset{w_1}{\to} e_{j+1}$ and $e_{j-1} \overset{w_1}{\to} e$ in $Pl(K)$. Then define $w_1' = \{e_j, e_{j+1}\}$. Similarly define a vertex w_2' corresponding to w_2. Replace $e = (w_1, w_2)$ by the edge (w_1', w_2'), which we will denote by $new(w_1, w_2)$. See Figure 4.

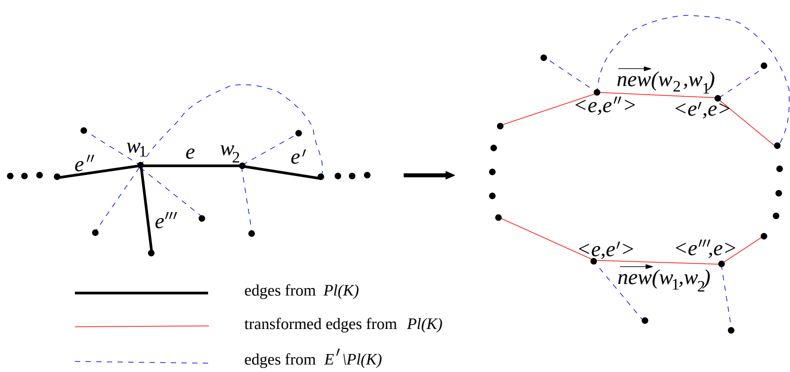


Fig. 4 Replacing an edge $e = (w_1, w_2)$ from $Pl(K)$ with a pair of new edges $\overrightarrow{new}(w_2, w_1)$ and $\overrightarrow{new}(w_1, w_2)$.

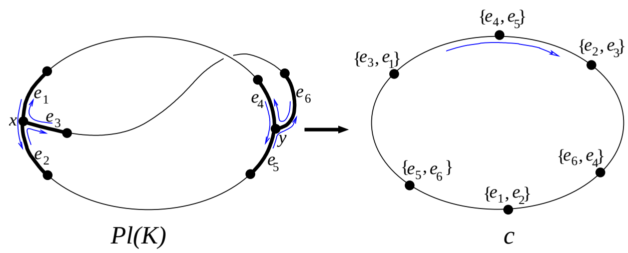


Fig. 5 Replacing $Pl(K)$ by a simple cycle C. The arrows show the direction of the face walk.

Finally we update the edge-orbits of the vertices incident to the new edges as follows. Let w be a vertex of $Pl(K)$ and let $< e_1 = (w, v_1), \cdots, e_k = (w, v_k) >$ be the counterclockwise permutation of the edges of K incident to w. For any pair of edges (w, v_i) and (w, v_j), $1 \leq i, j \leq k$, such that (w, v_j) is the first edge from $Pl(K)$ in a counterclockwise direction from (w, v_i), define the edge-orbit of the new vertex $w =< e_i, e_j >$ as follows:

$$< \overrightarrow{new}(w, v_i), new(w, v_{i+1}), \cdots, new(w, v_{j-1}), \overrightarrow{new}(w, v_j) > .$$

Denote by K' the resulting component, by $\bar{\mu}(K')$ its embedding, and by C the cycle corresponding to $Pl(K)$ (Figure 5).

In order to simplify notations, let $V_{Pl} = V(Pl(K))$, $V_c = V(C)$, and for any edge e let $new^*(e) = \overrightarrow{new}(e)$, if $e \in E(Pl(K))$, or $new^*(e) = new(e)$, otherwise. By the construction described above, we have the following.

Lemma 9 *The resulting component K', its embedding $\bar{\mu}(K')$, and the cycle C constructed by the transformation of $Pl(K)$ have the following properties:*

(a) $V(K) = V(K') \setminus V_c \cup V_{Pl}$, $E(K) = E(K') \setminus N(V_c) \cup N(V_{Pl})$;
(b) $\{N(v) \mid v \in V_{Pl}\} \setminus V_{Pl} = \{N(v) \mid v \in V_c\} \setminus V_c$;
(c) *If* $(v, w) \xrightarrow{v} (v, u)$ *in* $\mu(K)$, *then* $new^*(w, v) \xrightarrow{v^*} new^*(v, u)$ *in* $\bar{\mu}(K')$, *where* $v \in V_{Pl}$ *and* $v^* =< (v, w), (v, u) >$. *Intuitively, this property states that the transformation preserves the order of the edges in the edge orbits.*

Lemma 10 *The number of the faces of $\bar{\mu}(K')$ exceeds the number of the faces of $\mu(K)$ by at least one.*

Proof Let $((v_1, v_2), (v_2, v_3), \dots)$ be any face of $\mu(K)$. By Lemma 9 (c), $(new^*(v_1, v_2), new^*(v_2, v_3), \dots)$ will be a face of $\bar{\mu}(K')$. Hence, to each face of $\mu(K)$ there corresponds a face of $\bar{\mu}(K')$. Moreover, there is at least one face of $\bar{\mu}(K')$ that is not a face of $\mu(K)$. To see this, we use a simple counting argument. Each edge (v, w) of $Pl(K)$ is encountered a total of twice in all facial walks of $\mu(K)$, i.e., it is either encountered once in two different faces, or twice in a single face. (This is true for any edge in any embedding.) In K', however, there are two edges $\overrightarrow{new}(v, w)$ and $\overrightarrow{new}(w, v)$ corresponding to (v, w), each of which is encountered twice in all facial walks of $\bar{\mu}(K')$. Clearly, there will be a facial walk in $\bar{\mu}(K')$ that does not correspond to any facial walk in $\mu(K)$. $\square$

We show below (in Corollary 4) that there is actually exactly one more face in $\bar{\mu}(K')$ than in $\mu(K)$.

By computing the Euler characteristic of $\bar{\mu}(K')$ we prove the main result of this subsection.

Lemma 11 *The embedding $\bar{\mu}(K')$ is planar.*

Proof Let n_K, m_K, and f_K be the number of the vertices, edges, and faces of $\mu(K)$ and let n'_K, m'_K, and f'_K be those numbers for $\bar{\mu}(K')$. Since the genus of $\mu(K)$ is g_K, by (1)

$$n_K - m_K + f_K = 2 - 2g_K. \tag{9}$$

By Lemma 10,

$$f'_K \geq f_K + 1. \tag{10}$$

By Lemma 9 (a) and (b),

$$n'_K = n_K - n_{Pl} + |C|$$

$$m'_K = m_K - m_{Pl} + |C|.$$

Adding together the last three expressions and combining with (8) and (9), we get

$$n'_K - m'_K + f'_K \geq (n_K - m_K + f_K) - (n_{Pl} - m_{Pl}) + 1$$

$$= (2 - 2g_K) - (2 - 2g_K) + 2 = 2.$$

Hence, by (1), $\bar{\mu}(K')$ is planar. □

Corollary 4 *The number of the faces of $\bar{\mu}(K')$ exceeds the number of the faces of $\mu(K)$ by exactly one.*

Proof In the proof of Lemma 11 the inequality (10) cannot be strict, as otherwise $n'_K - m'_K + f'_K > 2$, which is impossible as the Euler characteristic cannot be greater than 2 (the genus cannot be negative.) □

That additional face in $\bar{\mu}(K')$ is exactly the face corresponding to the cycle C (Figure 5).

3.3.3 Transforming $\bar{\mu}(K')$ into a planar drawing of K with a small crossing number

Recall that the cycle C in K' corresponds to the subgraph $Pl(K)$ of K. In order to get K' from K, we replaced $Pl(K)$ with C. Replacing C with $Pl(K)$ will transform K' back into K.

Without loss of generality assume that the face corresponding to C is not the outer face. Remove all vertices from C and all of their incident edges. Denote by h the resulting face. Draw all vertices of $Pl(K)$ inside h. By Lemma 9 (b), all edges of K incident to a vertex in $Pl(K)$ will have both their endpoints inside h. Since there are no more than dn_{Pl} such original edges, they can be drawn inside h with no more than $(dn_{Pl})^2$ original crossings. We will need to slightly strengthen the last bound. Let n'_{Pl} be the number of vertices of Pl on levels in the interval (l^-_K, l^+_K), i.e., excluding vertices on levels l^-_K and l^+_K. Let $L_K = N(V(K)) \cap L_{i^*}$, where L_{i^*} was defined in (2). Then the number of original crossings is no greater than $(dn'_{Pl} + wt(L_K))^2$. That bound will be an improvement over the previous one in the case the vertices on levels l_K^- and l_K^+ have degrees significantly lower than d.

By Lemma 9 (a), the above operation transforms the embedding of K' into a planar drawing of K. Let $\bar{\mu}(K)$ denote the resulting drawing. We summarize the properties of that drawing in the following lemma.

Lemma 12 *$\bar{\mu}(K)$ is a drawing of K in the plane that has no more than $((dn'_{Pl} + wt(L_K))^2$ original crossings.*

We achieved our first goal, which is obtaining a planar drawing of K with a small number of crossings. What we still need to do is to route all the stubs of K to the outer face.

3.3.4 Routing the stubs of K to the outer face

Assign length 0 to all edges joining a pair of vertices on level l_K^- and assign length 1 to all other edges of K. The *length* $l(P)$ of a path P in K is defined as the sum of the lengths of its edges. We will make use of the following fact.

Lemma 13 *Let K be any component of G'. Between any pair of vertices of K there exists a path entirely in K of length no more than $2c_K^-(r-1) + c_K^- - 1$.*

Proof Let $B_1, \cdots, B_{c_K^-}$ be all bicomponents induced by the set of the vertices on level l_K^-. Then each vertex of K is a descendant of some vertex of B_j for some j. Let v and w be any two vertices of K. Construct a path $P = (v_1, \cdots, v_s)$ in K joining v and w. We will transform P into a path in K of length satisfying the lemma. Suppose v is a descendant of a vertex from B_i. Let j be the largest index for which v_j is a descendant of a vertex from B_i. Replace the subpath $(v_1, \cdots, v_j)$ with a simple path P_1 that uses forest edges only. Using the fact that edges joining vertices on the lowest levels l_K^- have length zero, the length of P_1 is no greater than $2(r-1)$. By the choice of j, the path $(v_{j+1}, \cdots, v_s)$ has no vertices that are descendants of a vertex from B_i. Using induction, the subpath $(v_{j+1}, \cdots, v_s)$ can be replaced by a path P_2 between v_{j+1} and v_s of length no more than $2(c_K^- - 1)(r-1) + c_K^- - 2$. Merging P_1, the edge (v_j, v_{j+1}), and P_2 results in a path between v and w of length at most $2c_K^-(r-1) + c_K^- - 1$. □

Lemma 14 *Each stub of K can be routed to the outer face h of $\bar{\mu}(K)$ with no more than $d(2c_K^-(r-1) + c_K^- - 1)$ crossings with original non-stub edges of K.*

Proof Let $s = (v, w)$ be a stub corresponding to an original edge, where v is the attached vertex of s, let u be the closest vertex from v on h, and let P be the path constructed by the procedure of Lemma 13 for v and u. Informally, s will be routed along a path "parallel" to P that avoids the vertices of P and that, for the portions of the path on level l_K^-, makes a "shortcut" inside the corresponding faces in order to minimize the number of intersections. Note that we need to pay special attention to the edges with endpoints on level l_K^- because such edges have length 0 and, hence, their number is not bounded by Lemma 13.

More formally, remove all edges e of K such that either (a) e is incident to a vertex of P on a level greater than l_K^-, or (b) e is incident to a vertex on level l_K^-, but e is not on P. The above operation creates a new face f that includes the

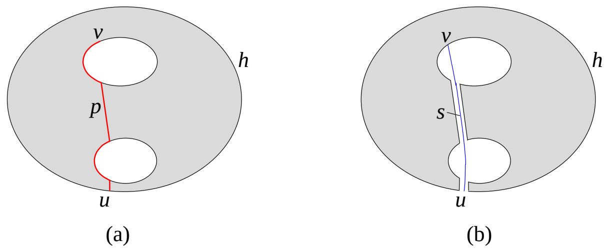


Fig. 6 (a) The path P between v and u. (b) Stub s is drawn in the face resulting after edges incident with P are deleted.

face h and all faces defined by the vertices on level l_K^- that contain the path P embedded inside it (Figure 6). Route s inside f avoiding vertices from P. Then s will intersect no more than $l(P)\cdot d$ original non-stub edges of G. □

The next lemma summarizes the results of this section regarding the resulting drawing of K.

Lemma 15 *The constructed drawing of K has less than $32(dg_K r)^2 + 3wt(L_K)^2 + 2dc_K^- r$ original crossings.*

Proof Crossings in the embedding of K may have occurred during the planarizing step (Subsection 3.3.3), or from stubs routed to the outer face (as in Lemma 14). By Lemma 12, the original crossings from the first type are no more than

$$((dn'_{Pl} + wt(L_K))^2,$$

which by Lemma 8 is no more than

$$(4d\, g_K(r-1) + wt(L_K))^2 < 32(rd\, g_K)^2 + 2wt(L_K)^2.$$

By Lemma 13, the number of original crossings of the second type are no more than

$$d(2c_K^-(r-1) + c_K^- - 1) < 2d\, c_K^-\, r,$$

excluding crossing between pairs of original stubs, which are no more than $wt(L_K)^2$. The lemma follows. □

We described an algorithm that draws a non-planar component of G' with a small number of crossings. When we apply that algorithm to all components of C_{np}, the total number of original crossings is estimated in the following lemma.

Lemma 16 *All non-planar components of G' can be drawn in the plane so that the unattached endpoints of all stubs are in the outer face of the drawing and the total number of original crossings is no more than $32(dgr)^2 + 4dgr + 3(m/r)^2$.*

Proof By Lemma 15 the total number of original crossings is bounded by

$$\sum_{K\in C_{np}} (32(rdg_K)^2 + 3wt^2(N(V(K)) \cap L_{i^*}) + 2dc_K^- r)$$
$$\le 32(dgr)^2 + 4dgr + 3wt^2(E(G) \cap L_{i^*}) \qquad \text{(by Lemma 6)}$$
$$\le 32(dgr)^2 + 4dgr + 3(m/r)^2 \qquad \text{(by (2))}\,.$$

□

3.4 Drawing m-planar components

This case is similar to the case of non-planar components in that it uses the fact that the sum of the numbers c_K^- for $K \in C_{mp}$ is of order $O(g)$. But, unlike C_{np}, in the case of C_{mp}, there is no need to planarize. We state the result in the following lemma.

Lemma 17 *All m-planar components of G' can be drawn in the plane so that the unattached endpoints of all stubs are in the outer face and the total number of original crossings is no more than $4dgr$.*

Proof Let K be any m-planar component. Draw K in the plane (with zero crossings) so that one of the cycles defined by the vertices on the lowest level is the outer face. Route each stub of K to the outer face of the drawing as described in the proof of Lemma 13. By Lemma 13, the drawing of K has less than $2dc_K^-r$ original crossings.

Draw by the same procedure all remaining m-planar components in the outer face of the drawing. By Lemma 7, the total number of crossings for all m-planar components is no more than

$$\sum_{K \in C_{mp}} 2dc_K^- r < 4dgr.$$

□

3.5 Drawing s-planar components

Unlike the previous two cases, in the case of s-planar components there is no analogue to Lemma 6 and Lemma 7 limiting the sum of the numbers c_K^-. In this case we use the fact that there is a single stub cycle adjacent to the lowest level of any s-planar component. Routing all stubs to the face corresponding to that cycle will be simpler than in the previous two cases.

Let K be any s-planar component and let C^- be the stub cycle for K. Denote by f^- the face defined by C^- and call it an outer face of $\mu(K)$. We can apply Lemma 14 to K substituting f^- for h, since both faces are on the lowest level of K. Since $c_K^- = 1$, then, by the lemma, each stub of K can be routed to f^- with no more than $2d(r-1)$ crossings with original non-stub edges of G. By (2), there are no more than $\lfloor m/r \rfloor$ stubs incident to the highest level of all components of $K(G')$, regardless of type. Hence, all stubs can be routed from the highest levels of all s-planar components to the faces corresponding to respective stub cycles with no more than

$$2d(r-1)\lfloor m/r \rfloor < 2dm$$

total crossings with original edges of G. Hence we have the following result for drawing s-planar components.

Lemma 18 *All s-planar components of G' can be drawn in the plane so that the unattached endpoints of all stubs are in the outer face and the total number of original crossings is no more than $2dm$.*

3.6 Reconnecting the embedded components

After all components of G' are drawn in the plane by applying the algorithms described in Subsections 3.3, 3.4, and 3.5, all the $wt(L_{i^*})$ original stubs will have their unattached endpoints in the outer face. Joining all pairs of original stubs into their parent edges so that no two stubs intersect more than once will produce at most $\binom{wt(L_{i^*})}{2}$ additional original crossings. This leads to the following statement, which is the main result of this section.

Theorem 1 *Any n-vertex graph of maximum degree d embedded in S_g can be drawn in the plane with $O(dgn)$ edge crossings.*

Proof By Lemmas 16, 17, and 18, the total number of original crossings from drawing individual components is no more than

$$32(dgr)^2 + 8dgr + 3(m/r)^2 + 2md = O((dgr)^2 + (m/r)^2 + md), \tag{11}$$

where m is the number of original edges of G. Assume that $g > 0$ and $d > 0$ as otherwise the theorem is trivially true. Choosing

$$r = \lceil \sqrt{m/(gd)}\, \rceil$$

in (11) and adding to the right-hand side of (11) the number of the original crossings resulting from joining the stubs in the final step, which is bounded by

$$wt^2(L_{i^*}) = O((m/r)^2),$$

the number of all original crossings is

$$O(dgm + md) = O(dgm).$$

Since $\mu(G)$ is a triangulation, we have the equality $f = \frac{2}{3}m$, where f is the number of the faces of $\mu(G)$. Consider the following two cases

1. $g = o(m)$. From the Euler formula (1)

$$n - 2 = m - f - 2g = m/3 - 2g \tag{12}$$

   and, since $g = o(m)$,

$$n - 2 = \Omega(m),\ m = O(n),\ O(dgm) = O(dgn),$$

   which proves the theorem in this case.
2. $g = \Omega(m)$. Let D be any drawing of G in the plane in which no two edges cross more than once (e.g., any straightline drawing). Then D has less than $m^2 = O(g^2)$ crossings. By (12), if $n > 1$,

$$\frac{m}{3} - 2g = n - 2 \geq 0$$

$$g \leq \frac{m}{6} \leq \frac{nd}{12}.$$

   Hence D will have no more than $O(g^2) = O(dgn)$ crossings.

□

Recall that $\mathrm{cr}_g(G)$ denotes the surface-g crossing number of G, i.e., the minimum number of crossings over all drawings of G on a surface of genus g. Theorem 1 shows that if $\mathrm{cr}_g(G) = 0$, then $\mathrm{cr}(G)$ cannot be very large. The next corollary generalizes this dependency by giving a relationship between the surface-g and planar crossing numbers for bounded degree graphs that are not necessarily of genus g.

Corollary 5 *Let G be any n-vertex bounded degree graph and let $0 < g = o(n)$. Then*

$$\mathrm{cr}(G) = O(\mathrm{cr}_g(G)\, g + gn).$$

Proof Draw G on S_g with $\mathrm{cr}_g(G)$ crossings. Replace every crossing by a new vertex. We get a new bounded degree graph G' of genus at most g and $\mathrm{cr}_g(G)+n$ vertices. By Theorem 1 we have

$$\mathrm{cr}(G') = O((\mathrm{cr}_g(G) + n)g).$$

Since

$$\mathrm{cr}(G) \leq \mathrm{cr}(G') + \mathrm{cr}_g(G),$$

the claim follows. □

3.7 Complete algorithm and complexity analysis

Here we describe the entire algorithm and analyze its complexity.

Algorithm SMALL_CROSSINGS_DRAW

Input: An n-vertex, d-degree graph G, an embedding $\mu(G)$ of G in S_g.
Output: A drawing of G with $O(dgn)$ crossings.

1. If $g > m = |E(G)|$, construct an arbitrary straightline drawing of G and exit.
2. Triangulate $\mu(G)$ assigning weight 0 to any new edge and weight 1 to any original edge of G.
3. If $d = 0$ or $g = 0$, construct any planar drawing of G and exit. Else set $r = \lceil \sqrt{m/(gd)}\, \rceil$. Choose any vertex t of G and divide the vertices of G into levels depending on their distances to t. Find a set L_i^* of edges such that (2) holds, as described in Section 3.1, and replace them by stubs, producing components of the following three types: (i) non-planar components, having induced genus greater than zero; (ii) m-planar components, having induced genus zero and at least two stub cycles; and (iii) s-planar components, having induced genus zero and at most one stub cycle.
4. For each component K, draw K in the plane applying one of the Steps 5, 6, or 7 below.
5. If K is non-planar, then apply the following steps.
   5.1. Construct a subgraph $Pl(K)$ of K such that (i) $Pl(K)$ contains at most $2g_K(r-1)$ vertices not counting the vertices on the highest and the lowest levels of K; (ii) converting $Pl(K)$ into a simple cycle C that is a face, denoted by f, of the new embedding as described in Section 3.3.2 transforms the embedding of K into an embedding of the updated graph, denoted by K', in S_0. Moreover, $Pl(K)$ and C have the same set, M, of edges joining them to K and K', respectively.

5.2. Draw K' in the plane with f as an outer face.
5.3. In order to transform K' back to K, remove C, draw the vertices of $Pl(K)$ in the resulting face, f', and draw all edges of M. Since both endpoints of any edge from M are on or inside f', intersections will occur only between pairs of edges from M.
5.4. Route any stubs of K to the infinite face of the drawing as described in the proof of Lemma 13 and join matching pairs of stubs into their parent edges.

6. If K is m-planar, then draw K in the plane with one of the stub cycles as an outer and route all stubs to that outer face as described in the proof of Lemma 17.
7. If K is s-planar, then draw K in the plane with the single stub cycle as an outer face and route all stubs of K to the outer face as described in the proof of Lemma 18.
8. Merge all drawings into a single drawing in the plane by merging the outer faces of the drawings of all components into a single outer infinite face and restore G by merging matching pairs of the remaining stubs (all located in the infinite face) into their parent edges.

Clearly Step 1 takes $O(|G|)$ time. Triangulating a face f in Step 2 takes time proportional to the number of the edges in f. Since each edge is encountered exactly twice in all facial walks of $\mu(G)$, triangulating all faces takes $O(m)$ time.

In Step 3, dividing the sets of vertices into levels can be done by a breadth-first search in $O(|G|)$ time. Finding a level i^* takes time proportional to the number of all levels, which cannot exceed n, and computing the set L_i^* takes additional time no more than $O(m)$. Finally, computing the induced genus of a component K requires computing the Euler characteristic of K, which takes $O|K|)$ time, and for finding the number of the stub cycles of K it is sufficient to scan all faces of $\mu(K)$, which also takes $O|K|)$ time. Hence Step 3 takes $O(|G|)$ time.

Steps 5.1 and 5.2 take $O(|K|)$ time, and Steps 5.3, 5.4, 6, and 7 can be implemented in time proportional to the number of the original crossings produced in these steps. Hence we have the following.

Theorem 2 *Algorithm SMALL_CROSSINGS_DRAW constructs a drawing of any n-vertex d-degree graph embedded in S_g satisfying Theorem 1 in $O(dgn)$ time.*

4 Lower bound

In this section we provide a matching lower bound for Theorem 1.

Theorem 3 *There is a constant α such that, for any positive integers n, $d < n$, and $g < dn$, there exists an n-vertex graph G of genus and degree not exceeding g and d, respectively, such that*

$$\mathrm{cr}(G) \geq \alpha dgn. \tag{13}$$

Proof We will show that there exists a graph of $\bar{n} = \Theta(n)$ vertices, degree $\bar{d} = \Theta(d)$, genus $\bar{g} = \Theta(g)$, and crossing number satisfying (13) for $\bar{n}$, $\bar{g}$, and $\bar{d}$ and some global constant α.

It is known [18,22] that any n-vertex graph H satisfies

$$\mathrm{cr}(H) \geq \frac{1}{40}\mathrm{bw}^2(H) - \frac{1}{16}\sqrt{\sum_{v \in V(H)} \deg^2(v)}. \tag{14}$$

In [21], it was a shown that for any $n > 0$, $d < n$, and $g < dn$, there exists an $\bar{n}$-vertex, $\bar{d}$-degree, $\bar{g}$-genus graph G, such that $\bar{n} = \Theta(n)$, $\bar{d} = \Theta(d)$, $\bar{g} = \Theta(g)$, and with bisection width

$$\mathrm{bw}(G) \geq \beta\sqrt{\bar{d}\bar{g}\bar{n}}, \tag{15}$$

for some absolute constant β. Suppose G is such a graph. By (14),

$$\mathrm{cr}(G) \geq \frac{\beta^2}{40}\bar{d}\bar{g}\bar{n} - \frac{d}{16}\sqrt{|V(G)|}$$

$$= \Omega(\bar{d}\bar{g}\bar{n}) - O(\bar{d}\sqrt{\bar{n}}) = \Omega(\bar{d}\bar{g}\bar{n}).$$

5 Extensions and generalizations

The bound in Theorem 2 on the time complexity of our algorithm is asymptotically optimal, if the algorithm is required to explicitly output all edge crossings. However, if a succinct encoding of the output is allowed, then a more careful implementation of the drawing phase of the algorithm can reduce the complexity to $O(|G|)$. For instance, instead of listing all intersections of an edge e with the edges incident to the vertices of a path $(v_0, \cdots, v_k)$ such that v_i is a parent of v_{i-1} in F_K, for $1 \leq i \leq k$, we need only to output the endpoints v_0 and v_K, since they uniquely determine that portion of the drawing. If $g = \Omega(n)$, then we need only to output the positions of the vertices, since they determine the drawing satisfying the theorem (the edges will be drawn as straight-line segments).

Pach and Tóth showed in [19] that their $O(c^g dn)$ upper bound on the crossing number of n-vertex d-degree graphs of genus g can be generalized to graphs of arbitrary degrees by substituting the product dn in their bound by the number $\sigma(G) = \sum_{v \in V(G)} \deg^2(v)$. The technique they use for this purpose is to replace any vertex v of G by a $\deg(v) \times \deg(v)$ 2-dimensional mesh M_v of vertices, where the edges incident to v appear in the same order around the boundary of M_v as they do around v. Clearly, the same technique, if used to modify the proof of our Theorem 1, will result in an upper bound of $O(\sigma(G)g)$. Such an approach, however, will increase the size of the graph, and hence the computation time, by a factor of $\sigma(G)/|G|$ (if a succinct encoding of the output is used), which can be as much as $\Omega(n)$. Such an increase in time can be reduced to only $O(\log n)$, while achieving the same bound on the crossing number, if, instead of replacing each vertex by a mesh, we assign a cost on each vertex equal to its degree and then use the technique from [8] to modify the definition of levels of the vertices of G in a way that takes into account the vertex costs.

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