Nuclear Physics B296 (1988) 431–444
North-Holland, Amsterdam

NUCLEON WAVE FUNCTIONS FROM LATTICE-GAUGE THEORIES
Measurements of baryonic operators

K.C. BOWLER[1], D. DANIEL[1], T.D. KIEU[1], D.G. RICHARDS[2] and C.J. SCOTT[3]

[1] *Physics Department, University of Edinburgh, Mayfield Road, Edinburgh EH9 3JZ, Scotland*
[2] *Theory Group, Argonne National Laboratory, 9700 South Cass Avenue, Argonne, IL 60439, USA*
[3] *Physics Department, The University, Southampton SO9 5NH, England*

Received 21 May 1987

We present measurements of the matrix elements of certain 3-quark operators that govern the short-distance and light-cone properties of the proton wave function obtained on an $8^3 \times 16$ lattice at $\beta = 5.7$ with Wilson fermions. Using these measurements we find the proton lifetime in the minimal SU(5) grand unified theory to be incompatible with the current experimental limits, in accord with another recent lattice calculation.

1. Introduction

The successful application of perturbative QCD over recent years to many hard processes has relied on the ability to absorb the "long-distance" non-perturbative effects into universal hadron distribution and fragmentation functions in the case of inclusive processes, and quark distribution amplitudes in the case of exclusive processes. The universality of these distributions allows different processes to be related to one another order by order in perturbation theory. The development of lattice gauge theories, however, has finally opened up the possibility of calculating these universal hadron distribution and fragmentation functions or quark distribution amplitudes directly within the framework of QCD. Calculations have been made of the weak interaction matrix elements [1], of the low moments of the quark distribution amplitude of the pion [2] and recently of one of the operators relevant to proton decay [3]. In an earlier paper [4], we developed the formalism for measuring the low moments of the quark distribution amplitude of the proton, expressed as matrix elements of the lowest twist 3-quark operators between the proton and the vacuum, as well as two 3-quark operators, of lowest dimension, that are required for the calculation of the proton lifetime in grand unified theories. The aim of this paper is to present measurements of the matrix elements of the operators of lowest dimension, and of the matrix element of the operator of lowest twist that

0550-3213/88/$03.50

sets the overall normalisation for the quark distribution amplitude in the proton. We begin with a brief review of the formalism developed in the earlier paper.

The leading twist component of the quark distribution amplitude of the proton is described by three wave functions $V(x_1, x_2, x_3)$, $A(x_1, x_2, x_3)$ and $T(x_1, x_2, x_3)$ where the x_i's are the longitudinal momentum fractions of the quarks in the proton. These three functions are not independent, but are related through the identity of two of the quarks and through isospin. Rather than studying the wave functions directly it is convenient to look at their moments defined by

$$V^{(n_1, n_2, n_3)} \equiv \int [\mathrm{d}x] x_1^{n_1} x_2^{n_2} x_3^{n_3} V(x_1, x_2, x_3) \tag{1.1}$$

with similar definitions for $A^{(n_1, n_2, n_3)}$ and $T^{(n_1, n_2, n_3)}$. These moments are related to the matrix elements of the leading twist contributions from the following local operators

$$\hat{V}_\tau^{(n_1, n_2, n_3)}(0) \equiv \left[(iD^{\mu_i})^{n_1} u(0)\right]^i C\gamma^\lambda \left[(iD^{\nu_i})^{n_2} u(0)\right]^j \left[(iD^{\zeta_i})^{n_3} (\gamma_5 d)_\tau(0)\right]^k \varepsilon^{ijk}, \tag{1.2a}$$

$$\hat{A}_\tau^{(n_1, n_2, n_3)}(0) \equiv \left[(iD^{\mu_i})^{n_1} u(0)\right]^i C\gamma^\lambda \gamma_5 \left[(iD^{\nu_i})^{n_2} u(0)\right]^j \left[(iD^{\zeta_i})^{n_3} d_\tau(0)\right]^k \varepsilon^{ijk} \tag{1.2b}$$

and

$$\begin{aligned} \hat{T}_\tau^{(n_1, n_2, n_3)}(0) \equiv {} & \left[(iD^{\mu_i})^{n_1} u(0)\right]^i C(-i\sigma_{\mu\lambda}) \left[(iD^{\nu_i})^{n_2} u(0)\right]^j \\ & \times \left[(iD^{\zeta_i})^{n_3} (\gamma^\mu \gamma_5 d)_\tau(0)\right]^k \varepsilon^{ijk}, \end{aligned} \tag{1.2c}$$

where C is the charge conjugation matrix, and

$$(iD^{\mu_i})^{n_1} \equiv (iD^{\mu_1})(iD^{\mu_2}) \dots (iD^{\mu_{n_1}}). \tag{1.2d}$$

Note that symmetrisation over Lorentz indices and the removal of traces is understood in the above. The leading twist contributions to the moments are then given by

$$\langle 0 | \hat{V}_\tau^{(n_1, n_2, n_3)}(0) | p \rangle = -f_N (p^{\mu_i})^{n_1} (p^{\nu_i})^{n_2} (p^{\zeta_i})^{n_3} p^\lambda N_\tau \cdot V^{(n_1, n_2, n_3)}, \tag{1.3a}$$

$$\langle 0 | \hat{A}_\tau^{(n_1, n_2, n_3)}(0) | p \rangle = -f_N (p^{\mu_i})^{n_1} (p^{\nu_i})^{n_2} (p^{\zeta_i})^{n_3} p^\lambda N_\tau \cdot A^{(n_1, n_2, n_3)}, \tag{1.3b}$$

$$\langle 0 | \hat{T}_\tau^{(n_1, n_2, n_3)}(0) | p \rangle = 2 f_N (p^{\mu_i})^{n_1} (p^{\nu_i})^{n_2} (p^{\zeta_i})^{n_3} p^\lambda N_\tau \cdot T^{(n_1, n_2, n_3)}, \tag{1.3c}$$

where N_τ is the proton spinor. The normalisation is chosen such that $V^{(0,0,0)} =$

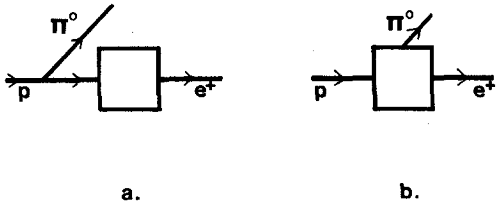

Fig. 1. Diagrams contributing to the amplitude for the decay $p \to \pi^0 e^+$ using the chiral lagrangian of refs. [5,6].

$T^{(0,0,0)} = 1$. ($A^{(0,0,0)}$ vanishes by symmetry.) In this paper we shall attempt to calculate the overall normalisation constant f_N (analogous to f_π in the case of pions).

Whereas the above explored the behaviour of the proton wave function at light-like separations, the process of proton decay is a short distance process, in that at least two of the quarks must be within a distance of $O(1/M_X)$. Of course it is not necessary for all of the quarks to be at such separations, and often the third quark will be just a spectator. However the chiral lagrangian approach [5,6] in the soft pion limit allows all decays of the form nucleon → anti-lepton + pseudoscalar, as illustrated in fig. 1, to be related to the matrix element of just two local operators

$$\langle 0|(O_\alpha)_\gamma|p\rangle \equiv \varepsilon^{ijk}\langle 0|\left(u_R^i C d_R^j\right)u_{L,\gamma}^k|p\rangle \equiv \alpha N_{L,\gamma}, \tag{1.4a}$$

$$\langle 0|(O_\beta)_\gamma|p\rangle \equiv \varepsilon^{ijk}\langle 0|\left(u_L^i C d_L^j\right)u_{L,\gamma}^k|p\rangle \equiv \beta N_{L,\gamma}. \tag{1.4b}$$

Furthermore, if the decay is mediated purely by the exchange of superheavy gauge bosons, as is the case with minimal grand unified theories, then the operators must involve both left- and right-handed fields so that the decay rate depends on the single parameter α. Thus, for example, in minimal SU(5) we have [6]

$$\Gamma(p \to \pi^0 e^+) = \tfrac{5}{4}\pi|\alpha|^2\alpha_{\mathrm{GUM}}^2 A^2(1+g_A)^2 \frac{m_p}{f_\pi^2 M_X^4}\left(1-\frac{m_\pi^2}{m_p^2}\right)^2, \tag{1.5}$$

where M_X is the mass of the superheavy gauge boson, α_{GUM} is the value of the coupling at the unification scale and A is the short-distance enhancement factor arising from QCD and weak interaction gauge boson exchange between some low energy hadronic scale Q and M_X. Though we adopt the chiral lagrangian approach in this paper we must stress that the validity of taking the soft pion limit is questionable in that the pion typically carries of the order of half of the momentum of the decaying proton.

The rest of this paper is laid out as follows. The next section will begin with the construction of the lattice operators whose matrix elements we wish to measure, and a description of how the matrix elements, as measured in the lattice at a spacing a, are related to their continuum counterparts in some specified renormalisation scheme, such as the Pauli-Villars(PV) scheme at a scale Q. We then list the correlators which we shall use to extract these matrix elements. Sect. 3 will contain details of the numerical simulation, followed by a presentation of our results and a discussion. We will focus in particular on the uncertainties in our results due to non-scaling effects, and in the size of the corrections arising when relating the lattice measurements to the continuum values. We will then look at the implications for the proton decay rate. Finally, sect. 4 contains a summary and our conclusions.

2. Construction and renormalisation of the lattice operators

The measurement of matrix elements in lattice gauge theories is complicated by the fact that a continuum operator belonging to a particular irreducible representation of the Lorentz group generally corresponds to an admixture of lattice operators lying in different representations of the hypercubic group of the lattice. The problem is particularly severe with the operators of eq. (1.3) possessing two or more derivatives where the construction of the continuum operators requires the subtraction of lattice operators diverging as the lattice spacing vanishes. Though this problem does not arise for the operators we are considering it is still necessary to evaluate the one-loop corrections in order to relate the operators as measured on the lattice to those as measured in the continuum using some (continuum) renormalisation scheme, which we choose to be the PV scheme at a scale Q. The changes in passing between different continuum schemes are small and will be neglected.

Computing the one-loop corrections to the operators O_α and O_β of eq. (1.4) in both the lattice and PV schemes for Wilson fermions at $r=1$ yields [4]

$$O_\alpha^{\mathrm{PV}}(Q)=O_\alpha^{\mathrm{L}}\left(\frac{1}{a}\right)\left\{1-\frac{\alpha}{4\pi}\left(\gamma\ln Q^2a^2+C_1^{\mathrm{L}}\right)\right\}-\frac{\alpha}{4\pi}C_2^{\mathrm{L}}O_\beta^{\mathrm{L}}\left(\frac{1}{a}\right)-\frac{\alpha}{4\pi}C_3^{\mathrm{L}}O_\gamma^{\mathrm{L}}\left(\frac{1}{a}\right), \tag{2.1a}$$

$$O_\beta^{\mathrm{PV}}(Q)=O_\beta^{\mathrm{L}}\left(\frac{1}{a}\right)\left\{1-\frac{\alpha}{4\pi}\left(\gamma\ln Q^2a^2+C_1^{\mathrm{L}}\right)\right\}-\frac{\alpha}{4\pi}C_2^{\mathrm{L}}O_\alpha^{\mathrm{L}}\left(\frac{1}{a}\right)+\frac{\alpha}{4\pi}C_3^{\mathrm{L}}O_\gamma^{\mathrm{L}}\left(\frac{1}{a}\right), \tag{2.1b}$$

where

$$\left(O_\gamma\right)_\tau\equiv\left\{u^iC\gamma_\rho\gamma_5d^j\right\}\left(\gamma_L\gamma^\rho u^k\right)_\tau\varepsilon^{ijk}, \tag{2.1c}$$

$$\gamma=-2,$$

$$C_1^{\mathrm{L}}=37.9,$$

$$C_2^{\mathrm{L}}=-3.2,$$

$$C_3^{\mathrm{L}}=-0.8 \tag{2.1d}$$

and

$$\gamma_{\mathrm{L}} = \tfrac{1}{2}(1 - \gamma^5). \tag{2.1e}$$

Thus, although there is a substantial correction in passing from the lattice operator measured at a spacing a to the continuum operator measured using the PV scheme at a scale $1/a$, the mixing between the operators, reflecting the chiral symmetry breaking induced by the Wilson parameter r, is small.

We have some freedom of choice as to which operator to use to measure f_{N}, and we work with

$$\hat{f}^{\nu}_{\gamma} \equiv \varepsilon^{ijk}\left[u^i C\gamma^{\nu}\gamma_5 d^j\right] u^k_{\gamma}, \tag{2.2}$$

which has isospin $\frac{1}{2}$ and therefore should interpolate well between the proton and the vacuum. Use of the Fierz identity yields

$$\hat{f}^{\nu}_{\gamma} = \tfrac{1}{2}\left[\hat{V}^{(0,0,0)} + \hat{A}^{(0,0,0)} - \hat{T}^{(0,0,0)}\right] + \text{operators of higher twist}, \tag{2.3}$$

so that

$$\langle 0|\hat{f}^{\nu}_{\gamma}|p\rangle = -\tfrac{3}{2} f_{\mathrm{N}} p^{\nu} N_{\gamma} + \text{higher twist terms}. \tag{2.4}$$

Computing the one-loop corrections to the leading twist part of $\hat{f}$ yields

$$\hat{f}^{\mathrm{PV}}(Q) = \hat{f}^{\mathrm{L}}\left(\frac{1}{a}\right)\left[1 - \frac{\alpha}{4\pi}\left(\gamma_0 \ln Q^2 a^2 + d\right)\right], \tag{2.5a}$$

where

$$\gamma_0 = \tfrac{2}{3},$$

$$d = 34.28. \tag{2.5b}$$

Thus the corrections in passing from the lattice measurement to the continuum are again potentially quite large.

The method employed in extracting the lattice matrix elements is the same as that used in measuring baryon masses; the time-sliced correlator of the operator of interest with some other interpolating operator is constructed. Thus for example, in determining α we construct

$$C_{\alpha}(t) = \sum_{\underline{x}} \langle 0|O_{\alpha}(t, \underline{x})_{\gamma}\left(\overline{O}_{\alpha}(0, \underline{0})\right)_{\gamma'}\gamma^0_{\gamma\gamma'}|0\rangle. \tag{2.6}$$

Inserting a complete set of states in eq. (2.6) and rotating into euclidean space yields

$$C_{\alpha}(T) = |\alpha|^2 \mathrm{e}^{-mT} + \text{terms that full faster with } T \tag{2.7}$$

for a lattice with Dirichlet boundary conditions in the time direction. In practice, we found it necessary to include two resonances in the right-hand side of eq. (2.7). The correlators corresponding to O_β and O_γ are constructed analogously.

In determining f_N we construct the following correlator

$$C_f(t) = \sum_{\underline{x}} \langle 0 | \tfrac{1}{2} \Big\{ \hat{f}_0(t, \underline{x}) \gamma_0 \bar{\hat{f}}_0(0, \underline{0}) \Big\}$$

$$- \tfrac{1}{6} \sum_{i \neq 0} \Big[\hat{f}_0(t, \underline{x}) \gamma_i \bar{\hat{f}}_i(0, \underline{0}) + \hat{f}_i(t, \underline{x}) \gamma_0 \bar{\hat{f}}_i(0, \underline{0}) + \hat{f}_i(t, \underline{x}) \gamma_i \bar{\hat{f}}_0(0, \underline{0}) \Big] | 0 \rangle . \quad (2.8)$$

The above combination of operators ensures that the non-leading twist components of the matrix element are removed. Inserting a complete set of states in eq. (2.8) and rotating into euclidean space yields

$$C_f(t) = \tfrac{9}{4} m^2 |f_N|^2 e^{-mT} + \text{terms that fall faster with } T. \quad (2.9)$$

3. Results and discussion

All measurements were made using gauge configurations generated in the quenched approximation in an $8^3 \times 16$ lattice at $\beta = 5.7$. Wilson propagators were calculated in 4-byte real arithmetic on the Edinburgh DAPs at three different quark masses corresponding to values of the hopping parameter $K = 0.1525$, 0.1575 and 0.1625, using periodic boundary conditions in the spatial directions, and Dirichlet boundary conditions in the time direction. The source was placed at the third time-slice. The critical value for the hopping parameter at this β value was determined in an earlier study [8] to be $K_c = 0.1695(7)$. The results of the present study are based on an analysis of 32 configurations, with successive configurations separated by at least 1200 sweeps to ensure minimal correlations between measurements. The correlators were fitted to a sum of exponentials and the following function minimized

$$\sum_{n_4} \left[\frac{\Delta(n_4) - C(n_4)}{|\Delta(n_4)| + |C(n_4)|} \right]^2, \quad (3.1)$$

where Δ and C are the measured and fitted values of the correlator respectively, and the sum is over an appropriate range of time-slices.

Errors were evaluated using the "jack-knife" procedure [7]; the parameters were calculated by fitting to the data averaged over all 32 configurations, the errors being estimated by performing fits to 32 ensembles of data, each obtained by averaging over 31 configurations. An indication of the quality of the fits was obtained by varying the range of time-slices to which the data were fitted.

In tables 1–3 we tabulate the 32-configuration averages of the correlators $C_\alpha(t)$, $C_\beta(t)$ and $C_f(t)$ respectively at time slices 5–15.

TABLE 1
32-configuration averages of correlator $C_\alpha(t)$

Time-slice	Bare quark mass		
n_4	m_1	m_2	m_3
5	$(2.60 \pm 0.07) \times 10^{-4}$	$(3.54 \pm 0.10) \times 10^{-4}$	$(4.64 \pm 0.14) \times 10^{-4}$
6	$(1.77 \pm 0.07) \times 10^{-5}$	$(2.80 \pm 0.12) \times 10^{-5}$	$(4.18 \pm 0.18) \times 10^{-5}$
7	$(1.72 \pm 0.10) \times 10^{-6}$	$(3.39 \pm 0.21) \times 10^{-6}$	$(6.26 \pm 0.46) \times 10^{-6}$
8	$(2.28 \pm 0.16) \times 10^{-7}$	$(5.86 \pm 0.51) \times 10^{-7}$	$(1.38 \pm 0.15) \times 10^{-6}$
9	$(3.68 \pm 0.36) \times 10^{-8}$	$(1.24 \pm 0.14) \times 10^{-7}$	$(3.65 \pm 0.49) \times 10^{-7}$
10	$(6.53 \pm 0.81) \times 10^{-9}$	$(2.79 \pm 0.42) \times 10^{-8}$	$(9.92 \pm 1.73) \times 10^{-8}$
11	$(1.16 \pm 0.17) \times 10^{-9}$	$(6.25 \pm 1.15) \times 10^{-9}$	$(2.63 \pm 0.67) \times 10^{-8}$
12	$(2.03 \pm 0.34) \times 10^{-10}$	$(1.40 \pm 0.31) \times 10^{-10}$	$(7.30 \pm 2.45) \times 10^{-9}$
13	$(3.78 \pm 0.74) \times 10^{-11}$	$(3.28 \pm 0.88) \times 10^{-10}$	$(1.81 \pm 0.92) \times 10^{-9}$
14	$(7.67 \pm 1.91) \times 10^{-12}$	$(9.08 \pm 3.18) \times 10^{-11}$	$(7.85 \pm 5.31) \times 10^{-10}$
15	$(1.68 \pm 0.64) \times 10^{-12}$	$(2.98 \pm 1.50) \times 10^{-11}$	$(3.99 \pm 3.53) \times 10^{-10}$

The protons contribution to $C_\alpha(t)$ and $C_\beta(t)$ (eq. (2.7)) are well exposed, and measurements of $|\alpha|$ and $|\beta|$ at each of the three-quark mass values are shown in fig. 2a and fig. 2b respectively. However the protons contribution to $C_f(t)$ (eq. (2.9)) is less well exposed, particularly at the smallest value of the quark mass. This is best illustrated in fig. 3, where we show the "effective" mass $M(n_4)$ defined by

$$M(n_4) \equiv \ln\left[\frac{C_f(n_4-1)}{C_f(n_4)}\right]. \tag{3.2}$$

TABLE 2
32-configuration averages of correlator $C_\beta(t)$

Time-slice	Bare quark mass		
n_4	m_1	m_2	m_3
5	$(3.09 \pm 0.10) \times 10^{-4}$	$(4.29 \pm 0.15) \times 10^{-4}$	$(5.77 \pm 0.20) \times 10^{-4}$
6	$(2.02 \pm 0.09) \times 10^{-5}$	$(3.23 \pm 0.16) \times 10^{-5}$	$(4.90 \pm 0.24) \times 10^{-5}$
7	$(1.84 \pm 0.11) \times 10^{-6}$	$(3.60 \pm 0.25) \times 10^{-6}$	$(6.66 \pm 0.50) \times 10^{-6}$
8	$(2.27 \pm 0.18) \times 10^{-7}$	$(5.73 \pm 0.56) \times 10^{-7}$	$(1.39 \pm 0.18) \times 10^{-6}$
9	$(3.52 \pm 0.37) \times 10^{-8}$	$(1.16 \pm 0.16) \times 10^{-7}$	$(3.88 \pm 0.67) \times 10^{-7}$
10	$(6.29 \pm 0.83) \times 10^{-9}$	$(2.72 \pm 0.43) \times 10^{-8}$	$(1.22 \pm 0.25) \times 10^{-7}$
11	$(1.14 \pm 0.17) \times 10^{-9}$	$(6.30 \pm 1.07) \times 10^{-9}$	$(3.31 \pm 0.87) \times 10^{-8}$
12	$(2.09 \pm 0.35) \times 10^{-10}$	$(1.50 \pm 0.31) \times 10^{-9}$	$(9.06 \pm 2.81) \times 10^{-9}$
13	$(3.90 \pm 0.75) \times 10^{-11}$	$(3.64 \pm 0.93) \times 10^{-10}$	$(3.34 \pm 1.43) \times 10^{-9}$
14	$(7.65 \pm 1.94) \times 10^{-12}$	$(8.75 \pm 3.17) \times 10^{-11}$	$(8.65 \pm 8.01) \times 10^{-10}$
15	$(1.47 \pm 0.50) \times 10^{-12}$	$(2.31 \pm 1.09) \times 10^{-11}$	$(3.62 \pm 4.25) \times 10^{-10}$

TABLE 3
32-configuration averages of correlator $C_f(t)$

Time-slice	Bare quark mass		
n_4	m_1	m_2	m_3
5	$(7.58 \pm 0.23) \times 10^{-4}$	$(9.82 \pm 0.33) \times 10^{-4}$	$(1.23 \pm 0.44) \times 10^{-3}$
6	$(4.37 \pm 0.21) \times 10^{-5}$	$(6.26 \pm 0.33) \times 10^{-5}$	$(8.33 \pm 0.46) \times 10^{-5}$
7	$(3.73 \pm 0.27) \times 10^{-6}$	$(6.34 \pm 0.55) \times 10^{-6}$	$(9.67 \pm 1.06) \times 10^{-6}$
8	$(4.39 \pm 0.41) \times 10^{-7}$	$(9.34 \pm 1.14) \times 10^{-7}$	$(1.75 \pm 0.29) \times 10^{-6}$
9	$(6.63 \pm 0.75) \times 10^{-8}$	$(1.81 \pm 0.27) \times 10^{-7}$	$(4.14 \pm 0.87) \times 10^{-7}$
10	$(1.15 \pm 0.16) \times 10^{-8}$	$(3.98 \pm 0.71) \times 10^{-8}$	$(1.05 \pm 0.27) \times 10^{-7}$
11	$(2.07 \pm 0.30) \times 10^{-9}$	$(9.14 \pm 1.74) \times 10^{-9}$	$(2.98 \pm 0.81) \times 10^{-8}$
12	$(3.57 \pm 0.59) \times 10^{-10}$	$(2.01 \pm 0.44) \times 10^{-9}$	$(9.97 \pm 3.75) \times 10^{-9}$
13	$(6.66 \pm 1.29) \times 10^{-11}$	$(5.13 \pm 1.40) \times 10^{-10}$	$(3.91 \pm 1.92) \times 10^{-9}$
14	$(1.37 \pm 0.36) \times 10^{-11}$	$(1.53 \pm 0.57) \times 10^{-10}$	$(1.87 \pm 1.05) \times 10^{-9}$
15	$(3.24 \pm 1.11) \times 10^{-12}$	$(4.96 \pm 2.34) \times 10^{-11}$	$(8.36 \pm 5.58) \times 10^{-10}$

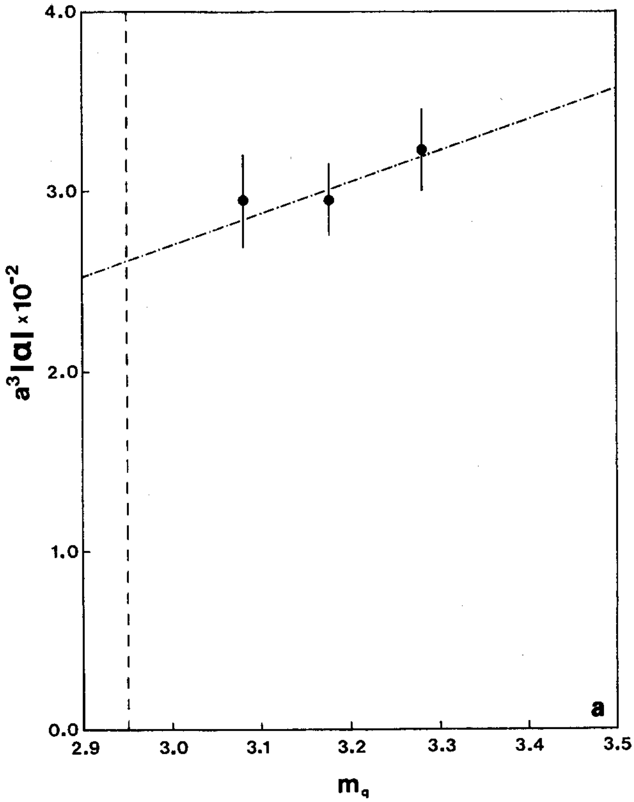

Fig. 2. Measurements of $|\alpha|, |\beta|$ at the three different values of the quark mass are shown in figs. 2a and 2b respectively. Also shown are the critical value of the quark mass (dashed line) corresponding to $m_\pi = 0$, and the extrapolation of the parameters to the physical value of the quark mass (dot-dash line).

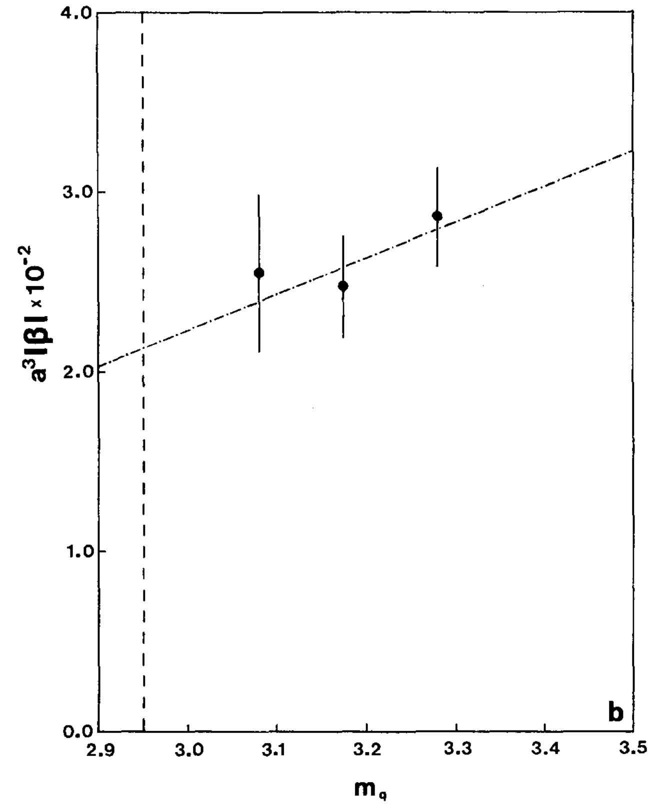

Fig. 2 (continued).

The contributions of the higher mass resonances in eq. (2.9) are substantially greater than those for $C_\alpha(t)$ and $C_\beta(t)$, and furthermore edge effects at the temporal boundary of the lattice infiltrate to smaller values of t. Indeed it is difficult to discern the protons contribution to $C_f(t)$ at the smallest value of the quark mass from fig. 3. Nevertheless in fig. 4, we show estimates of $|f_N|$ at each value of the quark mass, with the caveat that consistent fits at the lowest quark mass are obtained only over a limited range of time-slices. Finally we have also measured O_γ, introduced in eq. (2.1c), for a few configurations to ensure that it is not anomalously large.

We have extrapolated the data linearly in the quark mass in order to obtain estimates of the parameters at the physical quark mass, yielding

$$|\alpha|a^3 \simeq 2.6 \times 10^{-2},$$

$$|\beta|a^3 \simeq 2.1 \times 10^{-2},$$

$$|f_N|a^2 \simeq 1.5 \times 10^{-2}. \qquad (3.3)$$

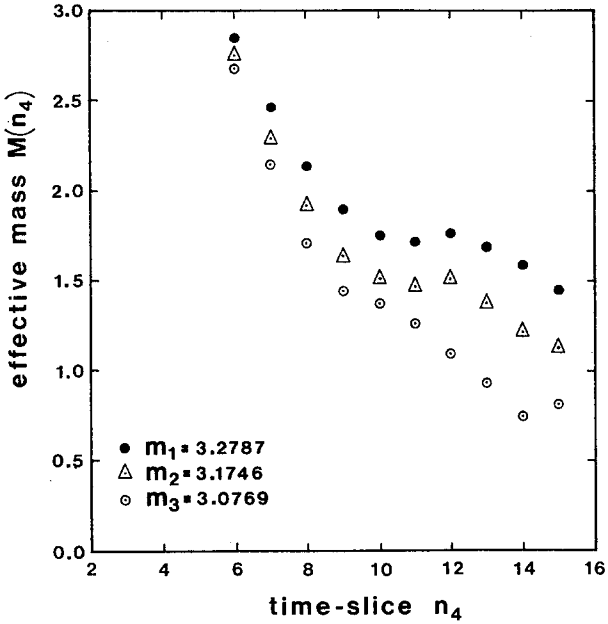

Fig. 3. A plot of the "effective" mass defined in eq. (3.2) versus the time slice for each value of the quark mass.

Turning to the question of what choice to make for the lattice spacing in eq. (3.3), we remark that unfortunately there is no unique value of the lattice spacing at $\beta = 5.7$ with Wilson fermions. In particular if the lattice spacing is chosen so as to ensure the correct value for the ρ-mass then the proton mass is overestimated [8]. We choose the lattice spacing so as to give the correct value for the proton mass, yielding

$$a^{-1} = 0.85 \pm 0.08 \text{ GeV}. \tag{3.4}$$

Using the central value of the lattice spacing yields

$$\left|\alpha^{\mathrm{L}}\left(\frac{1}{a}\right)\right| = 1.6 \times 10^{-2} \text{ GeV}^3,$$

$$\left|\beta^{\mathrm{L}}\left(\frac{1}{a}\right)\right| = 1.3 \times 10^{-2} \text{ GeV}^3,$$

$$\left|f_{\mathrm{N}}^{\mathrm{L}}\left(\frac{1}{a}\right)\right| = 1.1 \times 10^{-2} \text{ GeV}^2. \tag{3.5}$$

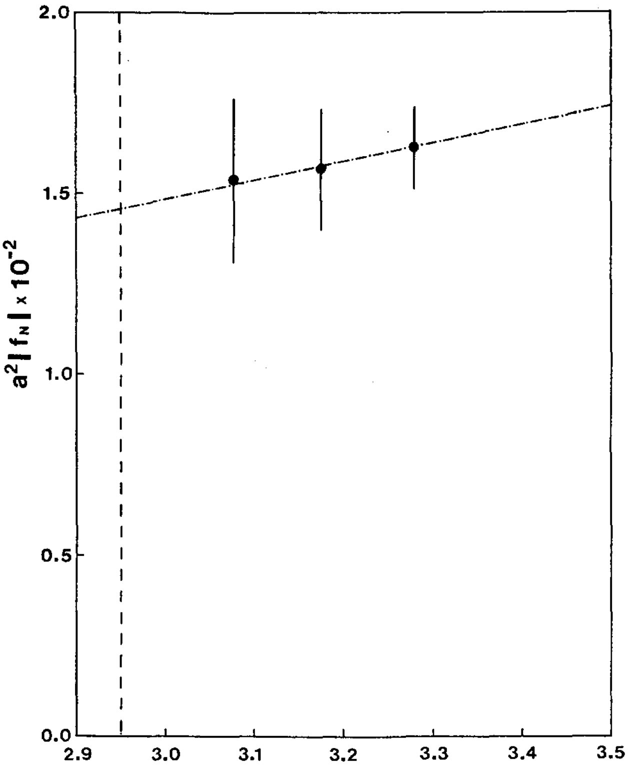

Fig. 4. Measurements of $|f_N|$ at the three values of the quark mass. Also shown are the critical value of the quark mass (dashed line) corresponding to $m_\pi = 0$, and the extrapolation of the parameters to the physical value of the quark mass (dot-dash line).

Other prescriptions for fixing the lattice spacing yield values for a^{-1} which are larger than that quoted in eq. (3.4) and will therefore tend to increase estimates of the proton decay rate.

We now address the question of the corrections arising in passing to the continuum theory, and begin by considering the lowest dimension operators. We see from eq. (2.1) that the correction due to the mixing with O_γ is small, and so we shall neglect it. Measurements of the nucleon mass by Bowler et al. [8] found that the operator

$$\varepsilon^{ijk}(u^i C d^j)\gamma_5 d^k$$

has a very small residue. Thus we expect α and β to have different signs so that

$$O^{\mathrm{PV}}_{\alpha,\beta}(Q) \simeq O^{\mathrm{L}}_{\alpha,\beta}\left(\frac{1}{a}\right)\left\{1 + \frac{\alpha}{4\pi}(2\ln Q^2 a^2 - C)\right\}, \tag{3.6}$$

where C is a number of order 40. The question as to what value to take for $\alpha/4\pi$ can only be answered by performing a higher order calculation. However C arises purely from the lattice calculation, and hence it is natural to use the coupling constant that is renormalized on the lattice at a spacing a. Even at the value of the inverse lattice spacing given by eq. (3.4), $\alpha/4\pi$ is only of the order of 1% since $\Lambda^{\mathrm{L}} \simeq \frac{1}{100}\Lambda^{\mathrm{MOM}}$ [9]. Thus the value of α or β measured in the continuum using the PV scheme at a scale $1/a$ is reduced from the value measured on the lattice at a spacing a by about 40%. Furthermore if we make the assumption that physics on the lattice corresponds to physics in the continuum at a scale 100 times higher (recall $\Lambda^{\mathrm{L}} \simeq \frac{1}{100}\Lambda^{\mathrm{MOM}}$) then we would take $Qa = 100$ in eq. (3.6) and the correction is reduced by one half. Thus we conclude

$$|\alpha| \simeq 1.3 \times 10^{-2}\ \mathrm{GeV}^3,$$

$$|\beta| \simeq 1.0 \times 10^{-2}\ \mathrm{GeV}^3, \tag{3.7}$$

where both quantities are quoted using the PV scheme at a scale $Q = 85$ GeV.

We now focus our attention on f_{N} and note from eq. (2.5) that the correction in passing from the lattice measurement to the Pauli-Villars at a scale $1/a$ is of the order of 35%. However if we adhere to our assumption that physics on the lattice at the spacing given by eq. (3.4) is equivalent to physics in the continuum at 85 GeV we find that the correction actually increases to about 40% yielding

$$|f_{\mathrm{N}}| \simeq 6.6 \times 10^{-3}\ \mathrm{GeV}^2. \tag{3.8}$$

The values quoted in eq. (3.7) are within the range of many other estimates using QCD sum rules or bag models. However a recent calculation by Hara et al. [3] on a $16^3 \times 48$ lattice with spacing $a^{-1} = 1.8$ GeV found a value somewhat larger than ours of $\alpha = 0.029\ \mathrm{GeV}^3$, reduced by, say, 20% to yield the PV renormalised quantity at $Q = 180$ GeV. We should point out that our prescription for relating the lattice measurements to the continuum differs from the one that they have adopted, with which we disagree in principle. Though the discrepancy between our measurements and theirs seems quite large, we must point out that there is no unique value of the lattice spacing at $\beta = 5.7$, and our value is chosen so as to put a lower bound on the parameters. Our value of $|f_{\mathrm{N}}|$ is actually rather larger than sum rule estimates [10] of $|f_{\mathrm{N}}| = (5.3 \pm 0.5) \times 10^{-3}\ \mathrm{GeV}^2$, but a discrepancy of only 20% is perhaps encouraging in view of the large systematic errors that also afflict sum rule calculations.

We conclude this section with a look at the implications of our results for the proton decay rate. Setting $\alpha = 1.3 \times 10^{-2}\ \mathrm{GeV}^3$, $\Lambda^{\mathrm{MS}} = 100$ MeV and truncating the short distance enhancement factor A at $Q = 85$ GeV, we find from eq. (1.5) that

$$\tau = (\mathrm{p} \to \pi^0 \mathrm{e}^+) \simeq 5.4 \times 10^{31} \times \left(\frac{M_{\mathrm{X}}}{10^{15}}\right)^4 \text{ years}. \tag{3.9}$$

Recent experimental limits on the proton decay rate yield [11]

$$\tau(\mathrm{p} \to \pi^0 \mathrm{e}^+) > 3.1 \times 10^{32} \text{ years}, \tag{3.10}$$

from which we conclude that for the minimal SU(5) model in the soft pion limit to be consistent with the known limits on the proton lifetime, the mass of the superheavy gauge boson at $\Lambda^{\mathrm{MS}} = 100$ MeV must satisfy

$$M_{\mathrm{X}} > 1.5 \times 10^{15} \text{ GeV}, \tag{3.11}$$

whereas recent estimates suggest [12]

$$M_{\mathrm{X}} = 1.3 \times 10^{14} \times (1.5)^{\pm 1} \text{ GeV}. \tag{3.12}$$

4. Conclusions

We have made measurements of the matrix elements of the 3-quark operators of lowest dimension relevant for proton decay, as well as of the 3-quark operator of lowest twist that determines the normalization of the proton wave function. At the value of β at which our measurements are made, we are not yet fully into the scaling region. Nevertheless, our estimates of α, β and f_{N} are encouragingly close to those obtained from the bag model and sum rule calculations. Furthermore, our estimate of the proton lifetime is unable to rescue the minimal SU(5) grand unified theory, in agreement with another recent lattice calculation.

Reliable numerical calculations will require simulations that are performed firstly at higher values of β, and secondly with the inclusion of dynamical quarks. Finally we need an understanding of finite size scaling effects.

We are grateful for many useful discussions with M. Grady and C.T.C. Sachrajda. This work was supported in part by the Science and Engineering Research Council under grant NG15908, and the US Department of Energy, Division of High Energy Physics, contract W-31-109-ENG-38. D.D. and C.J.S. wish to acknowledge SERC studentships and T.D.K. a Commonwealth Scholarship.

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